

NEW TEXTBOOKS FOR THE "NEW" MATHEMATICS

by *Richard P. Feynman*

As a member of the California State Curriculum Commission last year, I spent considerable time on the selection of mathematical textbooks for use in a modified arithmetic course for Grades 1 to 8 in California's elementary schools.

I have carefully read all of the books submitted by their publishers for possible adoption in California (18 feet of shelf space, 500 pounds of books!). Here I should like to describe and criticize these books in a general way, particularly with regard to the mathematical content — what it is we are trying to teach. I shall omit important matters, such as whether the books are written so that it is easy for the teacher to teach well from them, or the student to read them. Many of the books finally selected by the State for adoption do still contain some of the faults described below. This is because one could only select from what was submitted by the publishers, and few really good books were submitted. Also, budget limitations prevented adoption of most of the supplementary books that the Commission recommended in order to try to compensate for the faults of those basic books that were selected.

Why do we wish to modify the teaching of mathematics in the schools? It is only if we see this clearly that we can judge whether or not the new books satisfy the need. Most people — grocery clerks, for example — use a great deal of simple arithmetic in their daily life. In addition, there are those who use mathematics of a higher form — engineers and sci-

entists, statisticians, all types of economists, and business organizations with complex inventory systems and tax problems. Then there are those who go directly into applied mathematics. And finally there are the relatively few pure mathematicians.

When we plan for early training, then, we must pay attention not only to the everyday needs of almost everyone, but also to this large and rapidly expanding class of users of more advanced mathematics. It must be the kind of training that encourages the type of thinking that such people will later find most useful.

Many of the books go into considerable detail on subjects that are only of interest to pure mathematicians. Furthermore, the attitude toward many subjects is that of a pure mathematician. But we must not plan only to prepare pure mathematicians. In the first place, there are very few pure mathematicians and, in the second place, pure mathematicians have a point of view about the subject which is quite different from that of the users of mathematics. A pure mathematician is very impractical; he is not interested — in fact, he is purposely *dis*-interested — in the meaning of the mathematical symbols and letters and ideas; he is only interested in logical interconnection of the axioms, while the user of mathematics has to understand the connection of mathematics to the real world. Therefore we must pay more attention to the connection between mathematics and the things to which they apply

than a pure mathematician would be likely to do.

I hear a term called "new mathematics" used a great deal in connection with this program. That it's a new program of mathematics books is, of course, true, but whether it is wise to use "new," in the sense of very modern, mathematics is questionable. Mathematics which is used in engineering and science — in the design, for example, of radar antenna systems, in determining the position and orbits of the satellites, in inventory control, in the design of electrical machinery, in chemical research, and in the most esoteric forms of theoretical physics — is all really old mathematics, developed to a large extent before 1920.

A good deal of the mathematics which is used in the most advanced work of theoretical physics, for example, was not developed by mathematicians alone, but to a large extent by theoretical physicists themselves. Likewise, other people who use mathematics develop new ways to use it, and new forms of it. The pure mathematicians have in recent years (say, after 1920) turned to a large extent away from such applications and are instead deeply concerned with the basic definitions of number and line, and the interconnection of one branch of mathematics and another in a logical fashion. Great advances in this field have been made since 1920, but have had relatively little effect on applied, or useful, mathematics.

What we're after

I would consider our efforts to find new books and modify the teaching of arithmetic as an attempt to try to make it more interesting and easier for students to learn those attitudes of mind and that spirit of analysis which is required for efficient understanding and use of mathematics in engineering, science, and other fields.

The main change that is required is to remove the rigidity of thought found in the older arithmetic books. We must leave freedom for the mind to wander about in trying to solve problems. It is of no real advantage to introduce new subjects to be taught in the old way. To use mathematics successfully one must have a certain attitude of mind — to know that there are many ways to look at any problem and at any subject.

You need an answer for a certain problem: the question is how to get it. The successful user of mathematics is practically an inventor of new ways of obtaining answers in given situations. Even if the ways are well known, it is usually much easier for him to invent his own way — a new way or an old way — than it is to try to find an answer by looking it

up. The question he asks himself is not, "What is the right way to do this problem?" It is only necessary that he get the right answer.

This is much like a detective guessing and fitting his answer to the clues of a crime. In terms of the clues, he takes a guess as to the culprit and then sees whether that individual would be likely to fit with the crime. When he has finally suggested the right culprit, he sees that everything fits with his suggestion.

Any way that works

What is the best method to obtain the solution to a problem? The answer is, any way that works. So, what we want in arithmetic textbooks is not to teach a particular way of doing every problem but, rather, to teach what the original problem is, and to leave a much greater freedom in obtaining the answer — but, of course, no freedom as to what the right answer should be. That is to say, there may be several ways of adding 17 and 15 (or, rather, of obtaining the solution to the sum of 17 and 15) but there is only one correct answer.

What we have been doing in the past is teaching just one fixed way to do arithmetic problems, instead of teaching flexibility of mind — the various possible ways of writing down a problem, the possible ways of thinking about it, and the possible ways of getting at the problem.

This attitude of mind of a user of mathematics is, it turns out, also really the attitude of mind of a truly creative pure mathematician. It does not appear in his final proofs, which are simply demonstrations or complete logical arguments which prove that a certain conclusion is correct. These are the things that he publishes, but they in no way reflect the way that he works in order to obtain a guess as to what it is he is going to prove before he proves it. To do this he requires the same type of flexible mind that a user of mathematics needs.

In order to find an example of this, since I am not a pure mathematician, I reached up on the shelf and pulled down a book written by a pure mathematician. It happened to be *The Real Number System in An Algebraic Setting* by J. B. Roberts, and right away I found a quotation I could use:

"The scheme in mathematical thinking is to divine and demonstrate. There are no set patterns of procedure. We try this and that. We guess. We try to generalize the result in order to make the proof easier. We try special cases to see if any insight can be gained in this way. Finally — who knows how? — a proof is obtained."

So you see that mathematical thinking, both in



Richard P. Feynman, Richard Chace Tolman Professor of Theoretical Physics at Caltech

pure mathematics and in applied mathematics, is a free, intuitive business, and we wish to maintain that spirit in the introduction of children to arithmetic from the very earliest time. It is believed that this will not only better train the people who are going to use mathematics, but it may make the subject more interesting for other people and make it easier for them to learn.

In order to take this discussion of the character of mathematics away from the abstract, and to give a definite illustration, I'll choose material from the first and second years of school as an example; simply the problem of adding.

We suppose first that children can learn to count and that after a while they are very adept at counting. And now we wish to teach them addition. May we remark immediately that a child who can count well, say to 50 or 100, can immediately solve a problem such as $17 + 15 = 32$. For example, if there are 17 boys in the class and 15 girls, how many children are there in the class? The problem doesn't have to be given in the form of an abstract addition; it is simply a matter of counting the boys, counting the girls, and counting the class. We have, as a summary of the result: 17 boys plus 15 girls equals 32 children.

This method could be used to add any pair of whole numbers but, of course, it's a rather slow and cumbersome method for large numbers, or if a very large number of problems come up. Similar methods exist, such as having a set of counters or fingers and counting the things off with them. Another way is to count the numbers in the head. For example, after a while it is possible for a child to be able to add 3 to 6 by thinking to himself, 7, 8, 9. A more practical method is to learn by rote some of the simpler combinations, such as $3 + 6$, so that if they come up often it is not necessary to do the counting.

Counting large numbers — of pennies, for example — can be simplified by counting in groups instead of counting all the pennies. You can make little piles of five and count the piles of five; or better, you can make the piles of five into ten and then count the piles of ten first, and the number left over. Adding numbers could then be done more easily by adding the groups and the leftovers together.

Other ways of getting at addition facts (or doing additions, I should say) is to have a line on which the numbers are marked, or a thing like a calendar on which a whole series of numbers are written, one

after the other. Then, if you want to add 3 to 19, you start at 19 and count off three more spots along the line and come to 22. Incidentally, if these numbers are written as dots equally spaced along a line — called the number line — this becomes very useful later for an understanding of fractions and also of measurements; for inch-rulers and other things like thermometers are nothing but a number line written along the edge of the ruler. Therefore, putting the numbers on a line is useful, not only for learning addition in the first place, but also for understanding other types of numbers.

(Another special trick to remember, at a very elementary level, is that it is possible to determine which is the greater of two numbers without actually counting the numbers. If we have two rather large groups of things, it is easy to find out which group is larger by matching the things in pairs and seeing which group has objects left over. This is the way the number of molecules in different gases was first compared, by the way.)

Addition — the old way

In older books, addition is handled in a very definite way, without any variety of tricks or techniques. First we learn the simpler sums by drawing pictures of ducks — 5 ducks and 3 ducks, swimming, makes 8 ducks, and so forth — which is a perfectly satisfactory method. Then these numbers are memorized, which is again satisfactory. Finally, if the numbers are bigger than 10, a completely different technique is used. It is first explained how to write the numbers that are larger than 10 and then rules are given for the addition of two-column numbers, without carrying at first. It is not until the third grade that carrying can be done for the first time.

The dissatisfaction with the old text is not that any of the methods used to teach addition are unsatisfactory; they are all good. The trouble is, there are so few methods allowed that only a rigid and formal knowledge of arithmetic can result.

For example, a problem such as $29 + 3$ is not a legitimate problem for two years, for it is not given in the first and second grades, and the child is presumably unable to do that problem because, of course, for it you must carry. On the other hand, if you really understood what addition was, you could obtain the result of $29 + 3$ not very long after you have learned to count. Very early — in other words, in the first grade — you could do it by simply thinking 30, 31, 32.

It is true that this method is slow, but if no other method is available, then this is the method which ought to be used. It should be permitted to be used.

It should be one of the possible things that a child might do when he has to add by hook or by crook in a difficult problem. As he gets older he may increase his efficiency at doing the problems by using other methods, but it should be possible at the very earliest age to do addition problems with any reasonable numbers. There is really nothing different about adding 3 to 6 and adding 3 to 29; it is just that the technical, and generally more efficient, way we finally use when we get older is somewhat different.

A limited approach

In understanding the meaning of addition of two numbers — the meaning of sum, and how to get at it — there is no difference in the two problems. So the objection to the standard text is that only one method is given for making additions — namely, when the numbers are small, memorize them; when the numbers are larger, formally add them in a vertical column, two numbers together, and not carrying until after the second year. This is entirely too limiting for two years of study. If a child will not learn, or is unable to learn the formal rules, it should still be possible for him to obtain the result of some simple problem by counting or making a number line or by other technical methods.

In order to develop the kind of mental attitude which is required later, we should also try to give as wide a mathematical experience as possible. The sum should not appear always in the same form. There is no reason why every sum should be written 17 with a 15 underneath and a line drawn to obtain 32. A problem such as $17 + \quad = 32$, to fill in the blank, is a somewhat different variety but exactly the same type of question with numbers. So let the first-grade child cook up a way to obtain the answer to this problem. This is exactly the type of problem he will have to solve later if he becomes an engineer. I don't mean he will have to learn how to subtract. What I mean is that he has to deal with a new form of an old type of situation. The problem is to fill in a blank by any method whatsoever. However, when the blank is finally filled in, it must be correct.

We would not usually be interested, in engineering or in physics, in how a man obtained the result that 15 will go into the blank as long as he finally shows that 15 does work by simply adding the 15 and 17 and seeing if it comes out to 32.

(The only time we would be interested to know how he obtained the 15 is if this is the first time that such a problem has ever been done, and no one has ever known a way to do it previously — or if it seems likely that this type of problem will appear again and again in the future because of a new technical

development, and that we would like to have a more efficient method. Then it would be worth while to discuss the methods of obtaining the 15.)

So this problem, $17 + \text{a blank} = 32$ is an analogue of the general problem of applied mathematics, to find a way to fill in a blank number by any method whatsoever. It is a problem that could be given very early in the first year, leaving a freedom for the children to try to obtain the solution by any method they want, but of course not permitting wrong answers. The thing has to be checked out at the end.

Developing freedom

Here is another example of developing freedom, of a somewhat more complex form. Two times an unknown number $+ 3$ is 9. What is the unknown number? This is, of course, algebra, and there are very definite rules for solving such a problem — subtracting 3 from both sides and dividing by 2. But the number of algebraic equations that can be solved by definite rules is very small.

Another way is to try various numbers for the blank until one is found which fits. This way should be available to children at a very early age. In other words, problems should be put in many different forms. Children should be allowed to guess and to get at the answers in any way that they wish, in terms of those particular facts which they happen to memorize. Of course, it is necessary as time goes on for them to memorize the ordinary addition facts, the ordinary methods of making additions, multiplications, divisions, and so on, in addition to being allowed a freedom about the solution and the form of the various problems that are given to them.

Later, in more advanced work in engineering, when we have more complex algebraic equations, the only available method is, in fact, to try numbers. This is fundamentally a method that is of great power and will only have to be learned later by the student, or the engineer. The old teaching, that for every problem there is a definite fixed method, is only true for the simplest problems. For the more complex problems which actually arise there is *no* definite method, and one of the best ways to solve complex algebraic equations is by trial and error.

Another exercise which involves a greater degree of freedom is guessing a rule. This type of problem appears in more complex forms later, but a simple example, and a typical engineering or scientific problem, is the following: In a series of numbers 1, 4, 7, 10, 13, what is the pattern of rule by which they are being generated? The answer could be given in several ways. One is by adding 3 each time. Another is that the n th number is $3 \times n + 1$.

The key, then, is to give a wide variety of mathematical experience and not to have everything in a limited and stringent fixed form. This is not an argument about teaching methods. The point is not that it will then be easier to teach the regular arithmetic (although, for all I know, that may be so). The point is, it will be teaching a new subject in a sense — an attitude of mind toward numbers and toward mathematical questions which is precisely that attitude of mind which is so successful later in technical applications of mathematics.

It will not do simply to teach new subjects in the old way. For example, it has been recommended that numbers written in a different base than 10 be discussed in the early grades. This could serve to illustrate the freedom in mathematics to generalize, and help toward a deeper understanding of the reason behind the carrying rules in arithmetical operations. For this, a mention and an explanation with a few examples might delight some students. But if the matter is not understood by some of the slower members of the class, it is senseless to drill it in with interminable exercises, changing from one base to another. For such students, for whom a short exposure doesn't "take," more practice in the usual rules of base-10 computation is surely more sensible than *drilling* to perfection, calculation in base 5 and 12.

Words and definitions

When we come to consider the words and definitions which children ought to learn, we should be careful not to teach "just" words. It is possible to give an illusion of knowledge by teaching the technical words which someone uses in a field (which sound unusual to ordinary ears) without at the same time teaching any ideas or facts using these words. Many of the math books that are suggested now are full of such nonsense — of carefully and precisely defined special words that are used by pure mathematicians in their most subtle and difficult analyses, and are used by nobody else.

Secondly, the words which are used should be as close as possible to those in our everyday language; or, as a minimum requirement, they should be the very same words used, *at least*, by the users of mathematics in the sciences, and in engineering.

Consider the subject of geometry. It is necessary in geometry to learn many new words related to the mathematics. For example, one must learn what a triangle is, a square, a circle, a straight line, an angle, and a curved line. But one should not be satisfied solely to learn the words. At least somewhere, one should learn *facts* about the objects to which the

words refer, such as the area of the various figures; the relations of one figure to another; how to measure angles; possibly the fact that the sum of the angles of a triangle is 180° ; possibly the theorem of Pythagoras; or maybe some of the rules that make triangles congruent; or other geometrical facts. *Which* facts may be decided by those having more experience with curriculum, for I am not intending here to make any specific suggestions of what should be included and what not. I only mean to say that the subject of geometry, if it is taught at all, should include a reasonable knowledge of the geometrical figures over and above what the conventional names are.

Some of the books go a long way with the definition of a closed curve, open curve, closed regions, and open regions, and so on — and yet they teach no more geometry than the fact that a straight line drawn in a plane divides the plane into two pieces. At the end of some of these geometry books, look over to find, at the end of a long discourse, or a long effort at learning, just what knowledge of geometry has been acquired. I think that often the total number of facts that are learned is very small, while the total number of new words is very great. This is unsatisfactory. Furthermore, there is a tendency in some of the books to use most peculiar words — the words that are used in the most technical jargon of the pure mathematician. I see no reason for this.

It will be very easy for students to learn the new words when, and if, they become pure mathematicians and discourse with other mathematicians on the fundamentals of geometry. It is very easy indeed to learn how to use such words in a new way when one is older. A great deal of the objection that parents have to the so-called new mathematics may well be merely that it sounds rather silly to them when they hear their child trying to explain to them that a straight line is a “curve.” Such arguments in the home are absolutely unnecessary.

Precise language

In regard to this question of words, there is also in the new mathematics books a great deal of talk about the value of precise language — such things as that one must be very careful to distinguish a number from a numeral and, in general, a symbol from the object that it represents. The real problem in speech is not *precise* language. The problem is *clear* language. The desire is to have the idea clearly communicated to the other person. It is only necessary to be precise when there is some doubt as to the meaning of a phrase, and then the precision should be put in the place where the doubt exists. It is

really quite impossible to say anything with absolute precision, unless that thing is so abstracted from the real world as to not represent any real thing.

Pure mathematics is just such an abstraction from the real world, and pure mathematics does have a special precise language for dealing with its own special and technical subjects. But this precise language is not precise in any sense if you deal with the real objects of the world, and it is overly pedantic and quite confusing to use it unless there are some special subtleties which have to be carefully distinguished.

A fine distinction

For example, one of the books pedantically insists on pointing out that a picture of a ball and a ball are not the same thing. I doubt that any child would make an error in this particular direction. It is therefore unnecessary to be precise in the language and to say in each case, “Color the picture of the ball red,” whereas the ordinary book would say, “Color the ball red.”

As a matter of fact, it is impossible to be precise; the increase in precision to “color the picture of the ball” begins to produce doubts, whereas, before that, there was no difficulty. The picture of a ball includes a circle and includes a background. Should we color the entire square area in which the ball image appears or just the part inside the circle of the ball? Coloring the ball red is clear. Coloring the picture of the ball red has become somewhat more confused.

Although this sounds like a trivial example, this disease of increased precision rises in many of the textbooks to such a pitch that there are almost incomprehensibly complex sentences to say the very simplest thing. In a first-grade book (a *primer*, in fact) I find a sentence of the type: “Find out if the set of the lollypops is equal in number to the set of girls” — whereas what is meant is: “Find out if there are just enough lollypops for the girls.”

The parent will be frightened by this language. It says no more, and it says what it says in no more precise fashion than does the question: “Find out if there are just enough lollypops for the girls” — a perfectly understandable phrase to every child and every parent. There is no need for this nonsense of extra-special language, simply because that type of language is used by pure mathematicians. One does not learn a subject by using the words that people who know the subject use in discussing it. One must learn how to handle the ideas and then, when the subtleties arise which require special language, that special language can be used and developed

easily. In the meantime, clarity is the desire.

I believe that all of the exercises in all of the books, from the first to the eighth year, ought to be understandable to any ordinary adult — that is, the question of what one is trying to find out should be clear to every person. It may be that every adult is not able to *solve* all of the problems; perhaps they have forgotten their arithmetic, and they cannot readily obtain $2/3$ of $1/4$ of $1-1/3$, but they at least should understand that that product is what one is trying to obtain.

By putting the special language into the books, one appears to be learning a different subject and the parent (including highly trained engineers) is unable to help the child or to understand what the thing is all about. Yet such a lack of understanding is completely unnecessary and *no gain whatsoever* can be claimed for using unusual words when usual words are available, generally understood, and equally clear (usually, in fact, far clearer).

New definitions — and no facts

I believe that every subject which is presented in the textbook should be presented in such a way that the purpose of the presentation is made evident. The reason why the subject is there should be clear; the utility of the subject and its relevance to the world must be made clear to the pupil.

I would take, as an example, the subject of sets. In almost all of the textbooks which discuss sets, the material about sets is never used — nor is any explanation given as to why the concept is of any particular interest or utility. The only thing that is said is that “the concept of sets is very familiar.” This is, in fact, true. The idea of sets is so familiar that I do not understand the need for the patient discussion of the subject over and over by several of the textbooks if they have no use for the sets at the end at all.

It is an example of the use of words, new definitions of new words, but in this particular case a most extreme example, because *no facts whatever* are given at the end in almost all of the books. A zoo-keeper, instructing his assistant to take the sick lizards out of the cage, could say, “Take that set of animals which is the intersection of the set of lizards with the set of sick animals out of the cage.” This language is correct, precise, set theoretical language, but it says no more than, “Take the sick lizards out of the cage.” The concept of things which have common properties by being a member of two groups (such as the Chinese Communists, or of a larger number of groups such as East German refugee children) does involve intersections of sets, but



“You see, Daddy, this set equals all the dollars you earned; your expenses are a sub-set within it. A sub-set of that is your deductions.”

Drawing by Alan Dunn; © The New Yorker Magazine, Inc.

one does not use that language. No lack of precision results from this. And, besides, people who *use* mathematics in science, engineering, and so on, never use the long sentences of our imaginary zoo-keeper.

If we would like to, we can and *do* say, “The answer is a whole number less than 9 and bigger than 6,” but we do not have to say, “The answer is a member of the set which is the intersection of the set of those numbers which is larger than 6 and the set of numbers which are smaller than 9.”

It will perhaps surprise most people who have studied these textbooks to discover that the symbol U or \cap representing union and intersection of sets and the special use of the brackets $\{ \}$ and so forth, all the elaborate notation for sets that is given in these books, almost never appear in any writings in theoretical physics, in engineering, in business arithmetic, computer design, or other places where mathematics is being used. I see no need or reason for this all to be explained or to be taught in school. It is not a useful way to express one's self. It is not a cogent and simple way. It is claimed to be precise, but precise for what purpose?

Making the “new” mathematics worth while

In the “new” mathematics, then, first *there must be freedom of thought*; second, *we do not want to teach just words*; and third, *subjects should not be introduced without explaining the purpose or reason*, or without giving any way in which the material could be really used to discover something interesting. I don't think it is worth while teaching such material.