# Putnam Problems–Play by Play

For the past ten years on a Saturday in early December a Caltech team has fought for the school's honor (and individual glory) in national competition and has emerged victorious five times. It's not the football team — or the water polo team — but, what else, the mathematics team. It actually did not extend its winning streak this year but still came in a very respectable sixth in the William Lowell Putnam Mathematical Competition out of more than 200 teams from the United States and Canada. (Well, you can't win 'em all; USC didn't get to the Rose Bowl this year, either.) In the 41 years of the Putnam Competition only Harvard has won more times (9) than Caltech (8); besides the latest winning streak, the Beavers won in 1950, 1962, and 1964. In the past ten years only one other school has won more than once — Washington University (in St. Louis) in 1977 and 1980.

William Lowell Putnam, Harvard class of 1882, believed strongly in the merits of intellectual intercollegiate competition. In 1927 his widow created a trust fund to establish such a competition, which is now under the administration of the Mathematical Association of America. The first match was held in 1938. Among the early winners was Richard P. Feynman, then an undergraduate at MIT, now the Richard Chace Tolman Professor of Theoretical Physics and Nobel laureate.

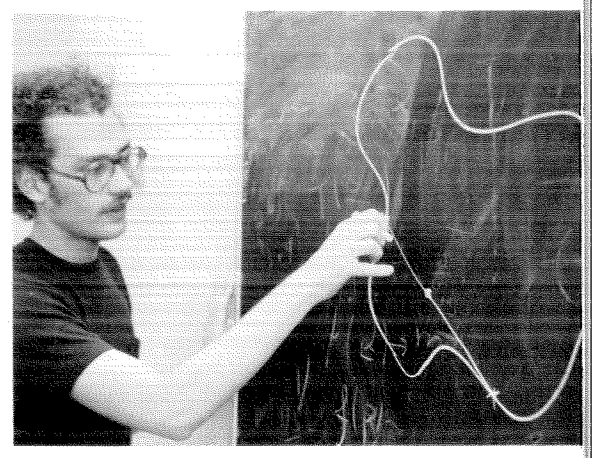
Nowadays more than 2,000 undergraduates enter the Putnam competition every year. On the first Saturday in December they spend three hours in the morning and three in the afternoon taking a written examination at their home schools. Everyone gets the same six problems in the morning and six more in the afternoon, made up each year by a committee of three mathematicians (often former Putnam winners). The problems are designed to test "originality as well as technical competence," which means simply that they tend to be very hard. The top grade last year was 61 percent, and only 43 students scored over 33 percent.

Contestants compete for cash prizes as well as personal glory (mathematicians don't have to worry about their amateur status). Prizes are awarded to the top five winners, whose ranking is not revealed, and to sixth through tenth place. To make things more interesting, some of the students are chosen in advance as team members representing their schools. Each team has three members, whose individual rankings on the test are added together to get the team score (the lower the better, just as in cross-country). The top five teams also win prizes for their respective mathematics departments.

Coach of the Caltech team is Gary Lorden, professor of mathematics, who doesn't really do much coaching. The Putnam team doesn't have to endure long hours of workouts — just one "how to solve it" session Lorden holds a few days before the exam. Although not requiring an extremely advanced or sophisticated knowledge of mathematics (freshmen enter as well as seniors), the examination presupposes familiarity with "subtleties beyond the routine solution devices" of differential equations and expects that "elementary concepts from group theory, set theory, graph theory, lattice theory, number theory, and cardinal arithmetic will not be entirely foreign to the contestant's experience." But ingenuity is the key to success.

The problems need to be fair as well as interesting, according to Lorden — fair in the sense that you shouldn't be able to arrive at a solution by luck, and interesting in that the solution should be intriguing to other mathematicians. Many of the problems also involve a kind of mathematics folklore — ideas that people who delight in solving this kind of problem usually seem to be familiar with. But it's not just a matter of a bagful of tricks either, says Lorden. "There aren't that many tricks to learn; mostly it's just being sort of clever and that is something that's hard to teach."

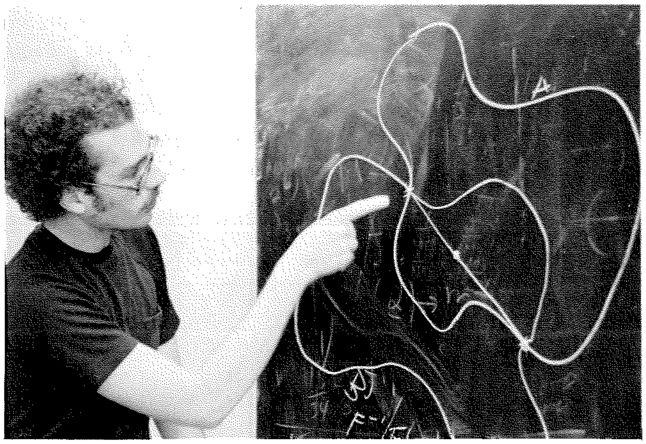
One of the cleverest of Caltech's recent entrants has been senior Peter Shor, a veteran of four years of the team. He came in sixth as a freshman, was ranked in the top five the following year, and then, with old age taking its inevitable toll a bit early, fell to 8th and 13th place respectively as a junior and senior. Shor had particularly good qualifications for the Putnam team; in high school he was a member of the U.S. team that won the 1977 Mathematics Olympiad against high school teams from 21 other countries. Any coach would welcome an Olympic winner! After graduation Shor plans to study math in graduate school at MIT.



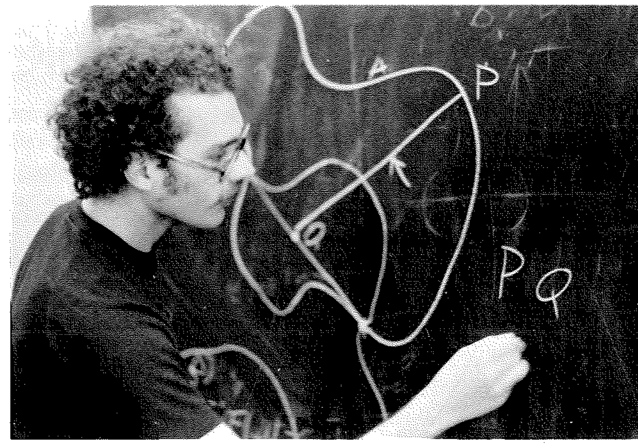
You want to find two points on any closed curve, says Peter Shor, so that a given point inside that curve is exactly in the middle of the line between those two points.

Although the answers to the Putnam problems are published annually in the American Mathematical Monthly, those answers are terse, to say the least, and not much help to the fairly casual observer, unless he happens to already know how to do it anyway. But Shor has picked out for *E&S* some of his favorite Putnam problems and explained how he arrived at the solutions. His explanations may not exactly make it all look easy, but at least they bring the Putnam Competition into range — like instant replay in slow motion — where the skills can be observed and appreciated by the great majority of us whose ingenuity falls somewhat short. Following is Shor's play-by-play description.

The first is problem B-4 from 1977: Let C be a continuous closed curve in the plane which does not cross itself and let Q be a point inside C. Show that there exist points $P_1$ and $P_2$ on C such that Q is the midpoint of the line segment $P_1P_2$.



If the curve rotated around the point crosses the original curve at one point, then it's also going to cross it at another point.




Some point P (designated $R_1$ in the text) on the original curve is going to be the farthest away from Q, so that PQ is a maximum distance.

You want to find two points on any closed curve so that a given point inside that curve is exactly in the middle of the line between those two points. Suppose you just rotate the entire curve 180 degrees in the same plane around the point Q. (Moving things around in a plane is a nice way of solving some problems. The Mathematics Olympiad had a lot of this sort of thing because we weren't supposed to know calculus yet.) Now, once you've done that, let's suppose that the rotated curve crosses, or intersects, the original curve. And if it crosses the original curve at one point, $P_1$, then it's also going to cross it at another point, $P_2$, on the other side of point Q (because you rotated it 180 degrees around Q). Since $P_1$ on the original curve corresponds to the position of $P_2$ on the rotated curve and $P_2$ on the original curve corresponds to $P_1$ on the rotated curve, then $P_1Q$ and $P_2Q$ have to be the same length. So Q is the midpoint of the line segment $P_1P_2$.

But that's not quite all. This is true *if* the curve crosses itself. You have to show that that *does* indeed happen; you have to show that when you rotate it 180 degrees around itself, there is some point where the rotated curve intersects the original one. That's not hard to do because you know that some point — let's call it $R_1$ — on this original curve is going to be the farthest point away from Q, so that $R_1Q$ is the maximum distance; there's another point, $R_2$, such that $R_2Q$ is the minimum distance from Q to the curve. If you want to be technical about it, this comes from something called compactness, which we don't want to worry about now. So this point $R_2$ — the closest one — must be inside the rotated curve, because the rotated curve can't have passed between this point and Q. If it did, there would have to be a point closer to Q, and there isn't because we defined $R_2$ as the closest. By the same reasoning, $R_1$ must be outside the rotated curve. And a curve connecting something inside and something outside has to cross somewhere. So we have shown that the original curve and the rotated curve do intersect. And we have already shown that if they do intersect, there must be two points that are equidistant from Q.

This next one is from the latest Putnam contest, problem A-4 from 1980:

(a) Prove that there exist integers a, b, c, not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$$

(b) Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that

$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$$

When you first look at it, it looks rather unusual — fairly bizarre really. You start looking at all these numbers — a, b, and c can be positive or negative, but each has to be less than 1 million, or $10^6$, and $a + b\sqrt{2} + c\sqrt{3}$ has to be less than $10^{-11}$, or .00000000001. Since this is very close to zero, one way to approach the problem is to find two numbers of this form that are very close together, so that when you subtract them, their difference is close to zero.

So let's say we have besides a, b, and c, $a_1$ and $b_1$ and $c_1$; if we subtract $a_1 + b_1\sqrt{2} + c_1\sqrt{3}$ from $a + b\sqrt{2} + c\sqrt{3}$, we end up with $(a-a_1) + (b-b_1)\sqrt{2} + (c-c_1)\sqrt{3}$, which is to be small. Just how small? If we can show that it's smaller than $10^{-11}$ we will have solved the first part of the problem. If a, b, and c, and $a_1$, $b_1$, and $c_1$ are positive integers less than $10^6$, then $a-a_1$ and $b-b_1$ and so on are less than a million, because if you subtract two ordinary positive integers less than 1 million, your answer can't be larger than a million; it could be negative but can't be bigger than $10^6$ or less than $-10^6$. Since $1 + \sqrt{2} + \sqrt{3}$ is less than 10, then the number $a + b\sqrt{2} + c\sqrt{3}$ must be less than $10^7$. And since these numbers are positive they lie between 0 and $10^7$.

We have a million choices for a, a million choices for b, and a million choices for c. If we plug them all in for $a + b\sqrt{2} + c\sqrt{3}$, we get $10^6$ times $10^6$ times $10^6$, or $10^{18}$ numbers of that form squeezed between 0 and $10^7$. If we space them all equally, then they would all be $10^7/10^{18}$, or $10^{-11}$, distance apart. And if we don't space them all equally, they're going to have to be closer than $10^{-11}$. There have to be at least two points in there that are closer together than $10^{-11}$ so that the difference when you subtract them is less than $10^{-11}$. Strictly speaking, even if they *were* all equally spaced, they would be closer than $10^{-11}$ because $1 + \sqrt{2} + \sqrt{3}$ is *less* than 10, which makes the interval actually less than $10^7$ and an average distance of less than $10^{-11}$.

And what if they're the same number? In this part of the problem that doesn't matter because subtracting them would give you zero, which is certainly less than $10^{-11}$. However, that brings us to the second part of the problem, which asks us to show that $|a + b\sqrt{2} + c\sqrt{3}|$ is greater than $10^{-21}$.

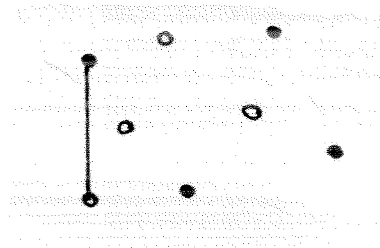
(The first part of this problem was supposed to be relatively easy. We're going to leave the second part for interested readers to tackle; it's more difficult and involves an entirely different idea and technique. It's not that Shor didn't figure it out; he did, and we will publish the solution in the next issue.)

And finally, problem A-4 from 1979: Let A be a set of 2n points in the plane, no three of which are collinear. Suppose that n of them are colored red and the remaining n blue. Prove or disprove: there are n closed straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors.

This is a connect-the-dots problem essentially. You've got n blue dots and n red dots and you have to connect blue ones to red ones with straight lines so that no lines cross. You have to show that this can be done no matter where the points are.

A good way to start is to put down a bunch of points and try to get a good idea. You look at them and think: Which points can you connect without interfering with any of the other possible connections? Which ones can you connect so that they won't come back to haunt you?

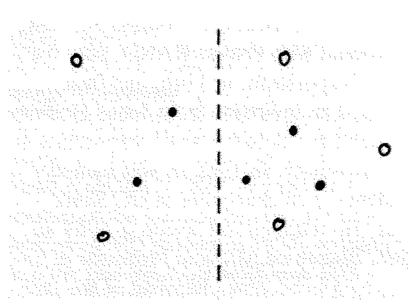
The obvious ones to connect are the ones on the outside. How do we define what "outside" means? If you connect all the points — every point with every other point regardless of color, there will be some connections located so that all the rest of the points will lie on one side of that line. (It won't work if there are three points on the same line, but the conditions state that no two lines have a point in common.) Now, if there are a red point and a blue point on the outside, then there has to be a red one next to a blue one someplace on the outside. So you connect these two, and that line will be on the outside and will not interfere with any subsequent lines that you draw. If you can connect two points on the outside, then you only have to work with the rest, which is a smaller number of each color, so you can just do the same thing again — reduced to a smaller problem — and so on. If you can do it with four of each color, then by induction you can do it with three, with two, and with one of each color.



However, to make this work we assumed there were a red point and a blue point somewhere on the outside to start with. The solution doesn't work if all the points on the outside are the same color — let's say blue. Since we can't connect two of the outside dots in this case, we have to think of something else. I remember staring at this for awhile before getting an idea — some way of measuring the points from left to right. We can put in a coordinate axis (and rotate things around if we have to so that this axis doesn't make any point directly above another, that is, the same left-to-rightness) and then label the points 1, 2, 3, and so on, starting from the leftmost point and numbering them to the rightmost point. No matter how we draw the axis, each of these extreme points has to be the same color, because all of the outside points are the same color in this case.

If you can draw a vertical line separating two consecutively numbered points somewhere in between, so that there is an equal number of red points and blue points on the left side (they will be the lower-numbered ones), then you can solve the problem for each side by the proof just mentioned. If the colors are equal on the left side, they will also be equal on the right side of such a line; the two sides don't have to be equal to each other.

We can take our coordinate axis and start moving it across the field of dots from left to right. The first dot we hit will have to be blue because we have stated that all the outside dots are blue. If we



"keep score" of the blue points minus the red points as we go across, then we would start out with 1 blue point and it would go up or down 1 with each point passed. For example, if the next point were blue, the "score" would go to 2. By the time we get to just before the last dot, we would have to be at $-1$, because the last dot will also have to be blue and we have to end up with zero. In going from 1 to $-1$, somewhere in between we would have had to cross zero, giving equal numbers of each color on either side.

It's not unusual in mathematics to have a hodgepodge of arguments like this. It may not be elegant, but it's legitimate, and sometimes the only way, to say that either this argument will work or, if it doesn't, then this other argument will work, and then prove that there aren't any cases where none of the arguments will work.

Now for all the Sunday-morning quarterbacks who have been saying, "Why, that's not so hard," throughout this article, here are some more Putnam problems to try out yourselves. Shor's answers will be published in the next issue of *E&S*.

From 1980:

*Problem B-3*

For which real numbers $a$ does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$?

(Express the answer in the simplest form.)

*Problem B-4*

Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leq i \leq 1066$. Prove there exist ten elements $x_1, \ldots, x_{10}$ of X such that every $A_i$ contains at least one of $x_1, \ldots, x_{10}$.

(Here $|S|$ means the number of elements in the set S.)

From 1978:

*Problem A-6*

Let n distinct points in the plane be given. Prove that fewer than $2n^{\frac{3}{2}}$ pairs of them are unit distance apart. (This may look simple, but it was one of the hardest problems that year.) □