Putnam Problem Solutions

A story in the June issue of *E&S* about the Putnam Mathematical Competition and Caltech's star competitor Peter Shor (BS '81) left some problems for readers to tackle. Here are Shor's answers.

Problem B-3, 1980

For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$?

(Express the answer in the simplest form.)

You can make the recursion easier by defining v_n's by

$$v_n = u_n \cdot 2^{-n}$$

u_n will be positive whenever v_n is positive.

Then $u_n = v_n 2^n$, and $v_0 = u_0 = a$.

The formula $u_{n+1} = 2u_n - n^2$

becomes $2^{n+1} v_{n+1} = 2 \cdot 2^n v_n - n^2$,

or $v_{n+1} = v_n - n^2 \cdot 2^{-n-1}$.

This gives

$$v_{n+1} = a - \frac{1}{2^2} - \frac{2^2}{2^3} - \frac{3^2}{2^4} - \ldots - \frac{n^2}{2^{n+1}}$$

v_n will be positive for all n if

$$\frac{1}{2^2} + \frac{2^2}{2^3} + \frac{3^2}{2^4} + \ldots \leq a$$

Now, if $x = \frac{1}{2}$ this is

$$x^2 + 2^2x^3 + 3^2x^4 + \ldots$$

To sum this series, consider

$$f(x) = 1 + x + x^2 + \ldots = \frac{1}{1 - x}$$

$$\frac{d}{dx} f(x) = 1 + 2x + 3x^2 + \ldots = \frac{1}{(1 - x)^2}$$

$$x\left(\frac{d}{dx} f(x)\right) = x + 2x^2 + 3x^3 + \ldots = \frac{x}{(1 - x)^2}$$

$$\frac{d}{dx}\left(x\frac{d}{dx} f(x)\right) = 1 + 4x + 9x^2 + \ldots = \frac{1 + x}{(1 - x)^3}$$

$$x^2\frac{d}{dx}\left(x\frac{d}{dx} f(x)\right) = x^2 + 4x^3 + 9x^4 + \ldots = \frac{x^2 (1 + x)}{(1 - x)^3}$$

and this series is what we want with $x = \frac{1}{2}$. So, the series converges to 3, and the answer is $a \geq 3$.

Problem B-4, 1980

Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leq i \leq 1066$. Prove there exist ten elements $x_1, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, \ldots, x_{10}$.

(Here |S| means the number of elements in the set S.)

We can find an element x_1 that is contained in more than half the sets A_i. If there were no such x_1, then $|A_1| + |A_2| + \ldots + |A_{1066}|$ would be less than or equal to $1066 \cdot \frac{1}{2}|X|$ because each x is in at most half the A_i's. But for each A_i, $|A_i| > \frac{1}{2}|X|$.

Consider only the sets A_i with x_1 not contained in A_i. There are at most 532 of these, and each of them contains more than half of the elements of X. Repeating the argument above, there is an x_2 contained in more than half of these 532 remaining sets, leaving at most 265 sets without x_1 or x_2. Repeating this ten times, you get $x_1, x_2, \ldots, x_{10}$, with each A_i containing one of them.

Problem A-6, 1978

Let n distinct points in the plane be given. Prove that fewer than $2n^{3/2}$ pairs of them are unit distance apart.

Draw a circle of radius 1 around each of the points p_i. Define e_i as the number of points p on the circle around p_i. $\frac{1}{2}\Sigma e_i$ is the number of pairs of points at distance 1, since each pair gives two points on the circles. We can assume $e_i \geq 1$ for all i, since otherwise we have a point unit distance from no points.

Each pair of circles has at most two intersections, so there are at most n(n-1) intersections. The point p_i occurs as an intersection $\frac{1}{2} e_i(e_i-1)$ times, so we have

$$n(n-1) \geq \Sigma \frac{e_i(e_i - 1)}{2} \geq \frac{1}{2} \Sigma (e_i-1)^2.$$

By the Cauchy-Schwarz inequality,

$$[\Sigma (e_i-1)]^2 \leq [\Sigma 1] [\Sigma (e_i-1)^2] \quad n \Sigma (e_i-1)^2.$$

So, combining the above equations,

$$[\Sigma (e_i-1)]^2 \leq n2n(n-1) < 2n^3, \text{ so}$$

$$\Sigma (e_i-1) \leq \sqrt{2}n^{3/2}$$

$$\frac{1}{2}(\Sigma e_i) \leq \frac{n + \sqrt{2}n^{3/2}}{2} < 2n^{3/2}$$

And finally, there's Problem A-4 from 1980, the first part of which was solved in the June issue, leaving (b), the harder part, for readers.

(a) Prove that there exist integers a, b, c, not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$$

(b) Let a, b, c be integers, not all zero and each of absolute value less than one million. Prove that

$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$$

In (b) you can multiply by conjugates to get rid of the square roots:

$$(a + b\sqrt{2} + c\sqrt{3})\, x\, (a + b\sqrt{2} - c\sqrt{3})\, x\, (a - b\sqrt{2} - c\sqrt{3})\, x\, (a - b\sqrt{2} + c\sqrt{3})$$
$$= a^4 + 4b^4 + 9c^4 - 4a^2b^2 - 6a^2c^2 - 12b^2c^2.$$

If a, b, and c are all integers, then this will be an integer, and if it's not equal to 0, it has to be of magnitude at least 1.

In the first part of the problem we showed that a number of this form had to be less than 10^7, so if the product of the conjugates is not 0, then we have $a + b\sqrt{2} + c\sqrt{3}$ times three factors that are less than 10^7 yielding something bigger than or equal to 1, and therefore

$$|(a + b\sqrt{2} + c\sqrt{3})|\, (10^7)^3 > 1$$
$$|a + b\sqrt{2} + c\sqrt{3}| > 1/10^{21}$$

and we're done.

Now all we have to do is show that the product of the conjugates cannot be equal to 0, and we can do this by proving that it's impossible for any of the four conjugate factors to be 0. If

$$a + b\sqrt{2} + c\sqrt{3} = 0, \text{ say,}$$

then

$$a + b\sqrt{2} = -c\sqrt{3}.$$

Now, squaring both sides, we get

$$a^2 + 2ab\sqrt{2} + 2b^2 = 3c^2$$
$$a^2 + 2b^2 - 3c^2 = -2ab\sqrt{2}.$$

If a and b are not 0, then this last equation could be used to express $\sqrt{2}$ as a quotient of integers, which is impossible since $\sqrt{2}$ isn't a rational number. So it only remains to consider the possibility that a = 0 or b = 0. If b = 0, then

$$a + c\sqrt{3} = 0,$$

which would make $\sqrt{3}$ a rational number unless c = 0. But if c is 0, then obviously a is 0, so a, b, and c are all 0, contrary to assumption. So the only remaining possibilities are in the case $a = 0, b \neq 0$.

In this case

$$b\sqrt{2} = -c\sqrt{3}$$

so we get

$$\sqrt{6} = -3c/b,$$

which is impossible because $\sqrt{6}$ is not rational, and this completes the argument.