THE THEORY OF GAMES

The study of games is tantamount to a study of human behavior in a given economic situation. Here, in simplified form, are some of the basic principles on which the theory of games is based.

by H. FREDERIC BOHNENBLUST

THREE CENTURIES AGO Pascal and Fermat investigated certain games of chance. Their attempt to find the laws governing chance moves led them to formulate the foundations of the theory which, in the course of time, evolved to the present day theories of probability and statistics.

Game theory can be said to have its origin in the work of Pascal and Fermat, but it is probably fairer to say that it dates back only about twenty years. At that time the mathematician John von Neumann undertook a systematic study of games. He gave a precise definition of what games came under his theory, and was particularly interested in investigating the interaction between the conflicting interests of the different players. What should be the behavior of a player? What can he be certain to achieve, and to what extent can his winnings be reduced by the plays of his opponents?

The significance of the theory of games goes far beyond a study of social games. When you consider that a game is a succession of moves, requiring each player at each stage to make a choice between several possible courses of action, and leading eventually to a certain pay-off for the different players, it is apparent that the study of games is tantamount to a study of human behavior in a given economic situation. However, serious difficulties of an economic—rather than mathematical—nature occur: it is difficult to isolate the economic situation, to delineate the possible courses of action and to evaluate the pay-off.

On these grounds the applicability of von Neumann's work to economics has been attacked. Even within a strict mathematical framework, the theory is far from complete. Nevertheless, the theory of games is here to stay and it is a rewarding effort to try to understand the simple principles on which it is based.

A complete definition of a game cannot be given here. It will suffice to mention that certain moves may be chance moves with known probabilities (the spinning of a roulette wheel, the shuffling of a deck of cards), and that others may be deliberately chosen from among the possible moves by the different participants in turn. The resulting choice, whether of a chance move or a deliberate move by a player, may be known to some, one, or even none of the players.

The rules of the game determine what will bring the game to an end, and what will be the actual pay-off to each participant. This outcome has a certain importance to each player. It is assumed that this importance can be measured by a number which represents the true interest of the player in the actual outcome. The game theory is concerned only in this final pay-off and will thus assume that each player is solely interested in

The cartoons on these pages are by Robert Osborn, and have been reproduced—by permission—from the book, STRATEGY IN POKER, BUSINESS AND WAR, *by John McDonald (W. W. Norton, New York, $2.50)* achieving as high a value for his final pay-off as he possibly can.

In an economic set-up, the actual outcome will involve generally varying amounts of different utilities, and the reduction to a numerical final pay-off function is a complex problem. (The reader should refer to the introductory chapter of the *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern for the background necessary to the construction of a numerical pay-off function.)

In order to simplify the study of games it is found convenient to try to reduce any given game to a certain standard form.

Standardized game

In theory, at least, all the possible ways of playing a given game can be listed explicitly, and each player can give, ahead of time, specific instructions to a double (or machine) describing how he intends to play the game. Such a set of instructions must be complete enough to provide for every contingency which can arise in the course of the game as a result of the chance moves and the plays of the opponents. The instructions replace the player, who need not even be present when the game is played.

A set of instructions is called a strategy, or more exactly, a pure strategy, for the player in question. A thoughtful friend can collect all the possible strategies of one player in a formidable tome, one to a page. Then the player need only glance through the book, point to a page, and let his deputy play the game.

If each player has listed his strategies, the game takes on the following simple form: each player chooses a page number from his book; this number is communicated to the umpire, but not to the other players; the umpire plays the game out according to the instructions received and the outcome is decided. Any game is thus reduced to a standard type. It is simple, theoretically, but complex practically, since even simple social games possess a high number of strategies.

The role of each player

The purpose of reducing a game to a standard type is to simplify the role of each player to making only one deliberate move—namely the choice of his page number. The game has not been changed in any way, but we have gained a clear understanding of the role of each player.

The job of the umpire is not yet fully automatic. In playing out the game he must still spin a roulette wheel (or shuffle a deck of cards) at each occurrence of a chance move. But the theory of probability teaches us that a succession of chance moves can be replaced by one chance move without changing the game. It teaches us more. This one chance move can be eliminated if the various pay-offs to which it leads are replaced by their expected value. For example, if the chance move is the throw of a perfect die with a pay-off of 2 dollars for the faces 1 or 2, and of 5 dollars otherwise, the expected value is equal to 2 x 1/3 plus 5 x 2/3 equals 4 dollars.

The game is now fully standardized. Let us consider, for example, a two-person game between two players, *A* and *B*. Let the rectangular array at the bottom of this page show the pay-off for the player *A*, and let a similar array be given for the second player.

In the illustration each player is assumed to have three strategies. The horizontal rows correspond to the strategies of *A*, and the vertical columns to those of *B*. The number 6 in the second row and first column tells, for instance, the pay-off to *A* should the first player *A* use his second strategy, and *B* his first strategy.

The sure thing

If from each row the least number is picked, the numbers —2, —1, 1, are obtained. The largest of these, 1, has a fundamental significance for *A*. He is certain to be able to achieve this amount. Yes, he need only decide to play his third strategy. Is he afraid that his nervousness or lack of a poker face will give him away? No, he can tell *B* beforehand what he is going to do and does not care whether *B* believes him or not. If no chance moves occur in the game, his certainty is absolute. In the presence of chance moves, the numbers in the array, as explained above, are not the actual pay-off, but are expected values, and it is only in the sense of an expected value that *A* is certain to achieve 1.

By removing himself one step further from the game, *A* actually can be certain to achieve more than 1. Rather than commit himself to a definite strategy, he can fix the probabilities with which the different strategies should be played, and let a chance move pick the strategy.

An assignment of probabilities to the strategies is

A \ B	FIRST	SECOND	THIRD
FIRST	0	2	-2
SECOND	6	-1	5
THIRD	4	1	3

called a mixed strategy. In the example considered, A can decide to play his first strategy with probability zero (i.e. not at all), his second strategy also with probability zero, and the third with probability one. This, of course, is equivalent to the definite choice of the third strategy. But he is not restricted to that choice.

He may, for example, decide to play his first strategy with probability 1/3, his second one with probability zero and his third one with probability 2/3. His expectation against the three defenses of B become 8/3, 4/3, 4/3, and, as an expected value, he is certain to achieve at least 4/3 instead of 1.

Naturally the player A can try to vary his mixed strategy until the guaranteed expected return is as high as possible. This so-called optimal mixed strategy can be determined mathematically. In the example under discussion it is the mixed strategy which has just been used.

Analyzing the zero-sum game

The analysis to this point has distinguished clearly the salient features of the game: the player chooses the mixed strategy; the mixed strategy determines—by a chance move—the pure strategy; and the pure strategy plays the game. In a two-person game each person has an array like the A that has just been discussed. A zero-sum game is one in which the array of the second player happens to be the negative of that of the first player. The analysis has proceeded far enough to analyze such games.

In these games the gain of one player is the loss of the other player. The interests of the players are strictly opposed; thus the interest of B is to minimize the numbers in the array of A. In the example, the mixed strategy of B which assigns the probabilities 0, 5/6, 1/6 to the pure strategies of B, leads to the expected values 4/3, 0, 4/3 against the three possible defenses of A. Thus B can be certain to prevent A from getting more than 4/3. It is remarkable that in both cases, for A and for B, the same number, 4/3, is obtained. This is not an accident for the numerical example considered. A fundamental result of the theory shows that for any two-person zero-sum game there exists a *unique number* v with the following property:

The first player A has at least one mixed strategy which guarantees him the expected value v against any defense of B. The second player B has at least one mixed strategy which guarantees him the expected value $-v$ against any defense of A.

Either player is willing to tell his opponent before the game the mixed strategy he is going to play. He cannot unwittingly betray himself by informing his opponent of the actual strategy which will be played, since he does not know it himself. It seems natural to define v to be the value of the game for the first player and $-v$ the value for the second one. If v is equal to 0 the game is fair; if v is positive, the first player is favored.

It may be well to summarize the reasons for calling v the value of the game. The player A can achieve v, but cannot reasonably expect more, since B can prevent his doing so. But why be reasonable? The unreasonable player achieves nothing beyond leaving himself open to a possible loss, particularly if his opponent should discover in time the mixed strategy or pure strategy he has chosen. However, a player may have good reason to believe his opponent too obtuse to play in an optimal way. He may be so convinced that the opponent will not use certain moves as to act on his conviction. Then the game is changed. The player should analyze the new game with these moves excluded.

The theory of two-person non-zero-sum games and of games between more than two players becomes more involved. The interests of the players are not in strict opposition, but may overlap. Some players may enter into coalitions. The coalitions act as one player and the joint winnings are distributed among the players of the coalitions according to some fixed agreement.

The complications are too numerous to be discussed in detail here. I will mention only that in three-person games the existence of "stable" coalitions can be established; stable, in the sense that the players in a coalition are afraid to break away for fear of ending in a still more unfavorable position. The dynamics by which stable coalitions are reached is, however, not understood. In games of more than three persons even the existence of stable coalitions is unknown. Many problems remain to be solved to make the theory complete. But even as it stands, it is captivating and contains many elegant and sound mathematical ideas which already have been applied to other fields.