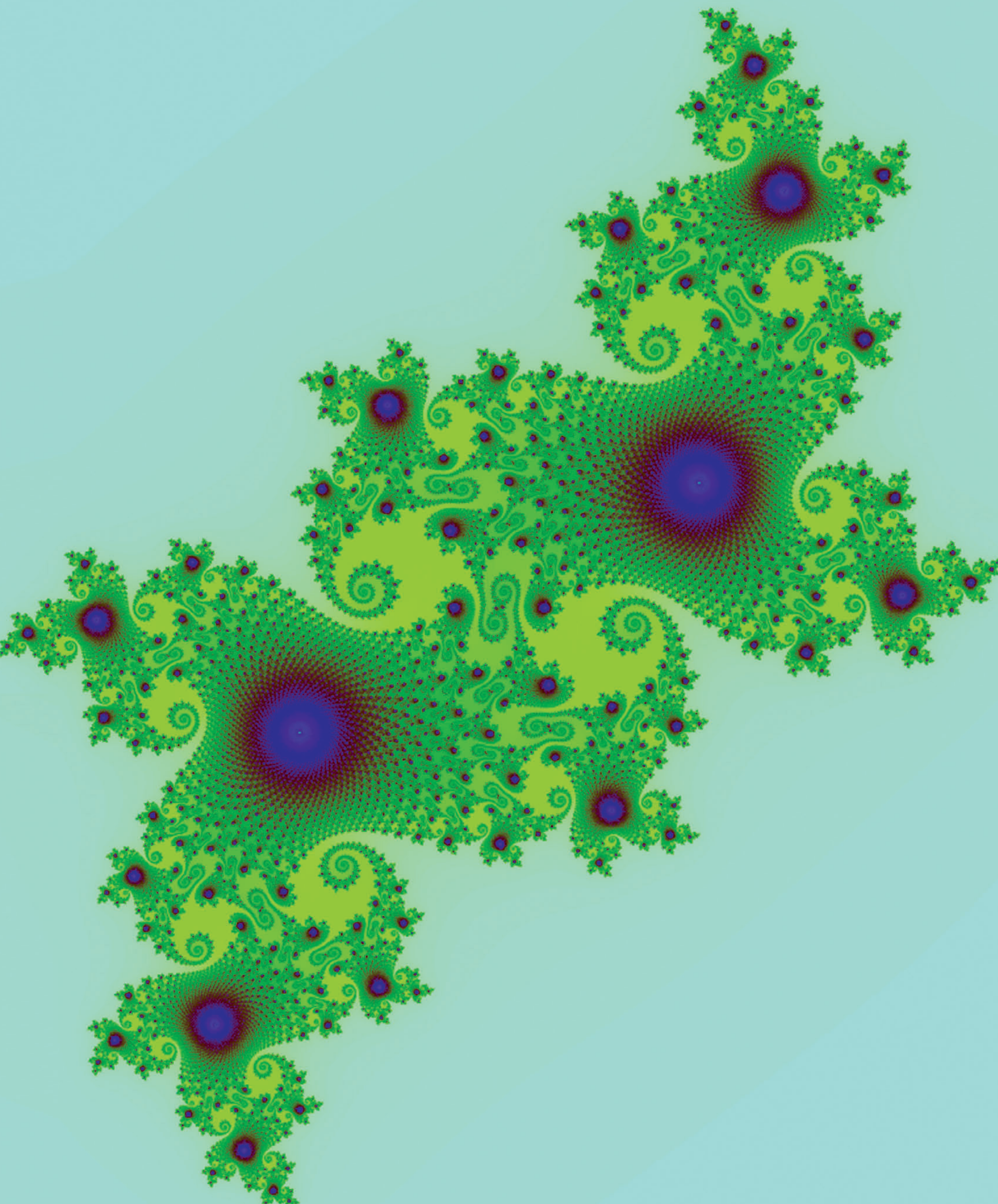


The Julia set at left is a fractal pattern. Because of their irregularity, fractals are what mathematicians call noncommutative spaces, which could be important to understanding the universe.

Unexpe

For Matilde Marcolli, physics and mathematics don't intersect so much as form a two-way street—which she travels in both directions, exploring the abstract world of math and pondering the nature of the cosmos.

The joke goes something like this: a physicist and a mathematician are staying at a hotel when, in the middle of the night, small fires break out in their rooms. The physicist wakes up and surveys the scene. She grabs the hotel note pad and pen and does some quick calculations. After determining exactly how much water is needed to extinguish the fire, she fills up the ice bucket with water and pours it over the flames. In the other room, the mathematician wakes up and sees the fire. She gets up and looks over at the faucet and notices the ice bucket. "Ah ha! There's a solution to the problem!" she says. Satisfied, she climbs back into bed.

There are many versions of this joke, all of which seem to come at the expense of the mathematician, and it illustrates a certain divide that has plagued the two professions. Although they both work with complicated equations and share areas of research, mathematicians and physicists are seen as different breeds with different passions. For a pure mathematician, the beauty of a theorem is an end in itself, a greater truth that transcends physical reality. The fact that it is possible to extinguish the fire is enough to fulfill the mathematician's desires—any relevance to the real world is incidental. "Mathematics possesses not only truth, but supreme beauty," Bertrand Russell said. But for a physicist, math is the language of nature, a mere tool for understanding how the universe works. Richard Feynman, embodying the brash physicist, said, "I love only nature, and I hate mathematicians."

cted Connections

By Marcus Y. Woo

But math and mathematicians have always been inextricably tied to physics and physicists. Isaac Newton invented calculus to describe the laws of motion. Albert Einstein used a branch of pure mathematics called differential geometry to formulate his theory of gravity as warped space and time. You can't do physics without math, and regardless of cultural and intellectual differences—real, perceived, or manufactured—the two are committed to each other, in sickness and in health, till death do them part.

Matilde Marcolli is well versed in the relationship between math and physics. She studied physics at the University of Milan before going to the University of Chicago for a PhD in mathematics, and is now a professor of mathematics at Caltech. She thrives on taking seemingly unrelated ideas from physics and applying them to solve math problems—and vice versa. As a theoretical physicist, she develops new models of the universe and possible theories of quantum gravity, the so-called theory of everything. For Marcolli, math and physics don't intersect as much as form a two-way street—and she goes in both directions.

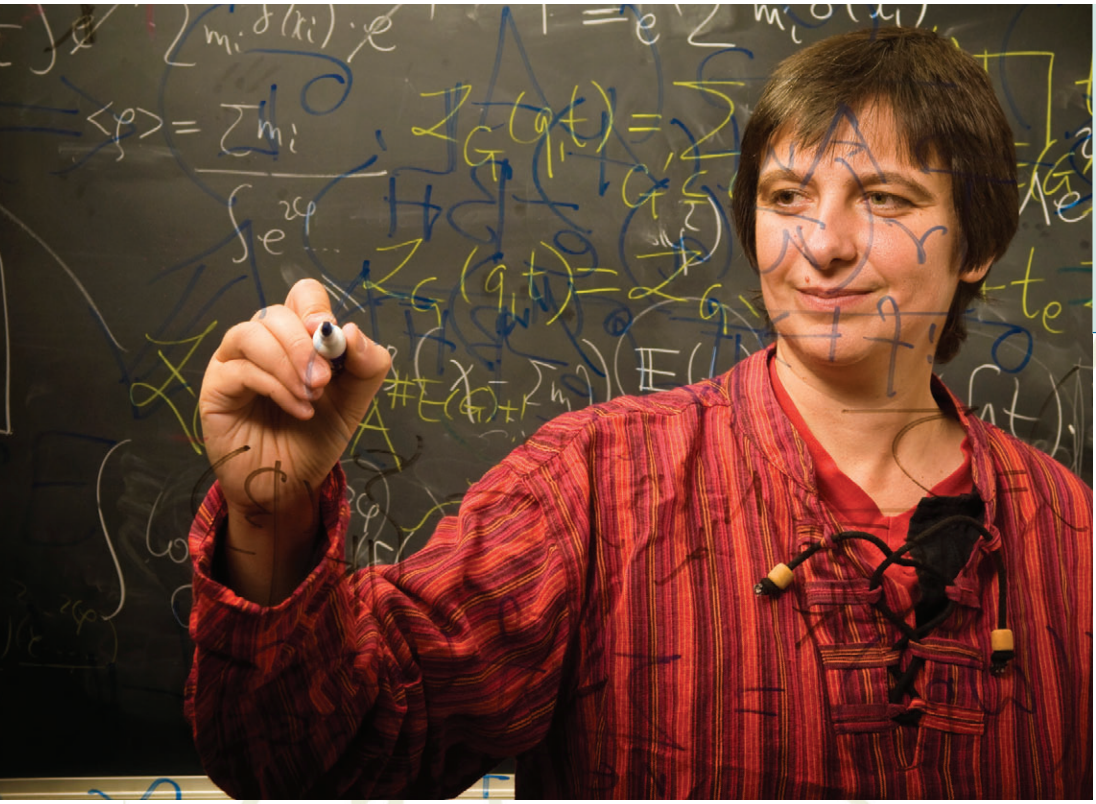
"One of the most exciting things in science is seeing unexpected connections between different things," she says. "At any given time, you're just looking at a very small detail of this great big picture, and you try to connect as many dots as you can." So does she consider herself a physicist or a mathematician? "Depends on the day," she quips.

On the days when Marcolli is mulling mathematics, she works on problems in areas such as number theory, which is the study of numbers and their properties. Although number theory has some emerging applications—in cryptography, for instance—it's about as pure as math can get. But with Gunther Cornelissen from the University of Utrecht in the Netherlands, Marcolli has recently applied ideas grounded in the physical world—albeit in quantum physics—to solve a problem in the even more abstract world of numbers, involving objects called number fields.

NUMBER FIELDS FOREVER

Most of us think of numbers as mere quantities, a way to represent how many apples you have in your bag.

One place number theory meets the real world is in the Fibonacci series (0, 1, 1, 2, 3, 5, 8, 13, 21, 34 . . .) where every number is the sum of the two that preceded it. The number of florets in each spiral on this head of Romanesco broccoli is a member of the series. The resulting pattern forms a fractal.



Regardless of whether she's a physicist or a mathematician, Matilde Marcolli immerses herself in a sea of equations.

But for a mathematician, they're like organisms with their own behaviors and characteristics, and, like organisms, numbers can be classified according to their properties and the operations one can perform with them. One way to classify numbers is by constructing a field, a set of numbers that satisfy certain rules. An example is the field of rational numbers—numbers that can always be written as a ratio of two integers and that obey rules like addition, subtraction, division, and multiplication.

You can generate a *number* field by extending the field of the rational numbers. To extend the field, you can include certain kinds of irrational numbers, such as $\sqrt{2}$. (For the mathematically inclined, these numbers must be solutions of polynomial equations with integer coefficients.) The extended field still contains all rational numbers, but also $\sqrt{2}$, and any combination thereof, such as $\sqrt{2} + 1$.

It turns out that you can use the numbers in a field to define various functions. As you might recall from high-school math, some simple functions include sine and cosine, where you put in one number and out comes another. One famous function is the Riemann zeta function, which is closely associated with prime numbers—integers that are only divisible by one or themselves. Each number field has its own prime numbers and a corresponding generalization of the Riemann zeta function, called the Dedekind zeta function.

Number fields and functions come from two completely different areas of mathematics, Marcolli explains, since number fields are discrete quantities while functions are continuous objects. The fact that the two have anything to do with each other is just another instance of connecting bits and pieces to make sense of the larger mathematical picture.

Marcolli and Cornelissen looked at a set of functions that includes the Dedekind zeta function, trying to understand how well these functions reveal the properties of the corresponding number fields. Although you can start with a number field and build functions from it, it's not obvious that you can go the other way and determine the corresponding number field solely from a set of functions. In other words, it's easy to take two-by-fours, nails, Sheetrock, and paint and build a wall. But if you just look at a finished wall, you can't really tell what components were used to construct it.

Mathematicians have long known that the Dedekind zeta function by itself was not enough to determine the field. But to their surprise, Marcolli and Cornelissen discovered that when you have the Dedekind zeta function *and* these other functions, you actually could characterize the corresponding number field. "These functions know everything there is to know about the field," Marcolli says.

And here's where the physics comes in: these functions describe the so-called equilibrium states of a quantum system, which consists of a collection of particles, such as a bunch of electrons. This type of system behaves like a collection of tiny magnets, Marcolli explains. At low temperatures, their poles align and point north. But if you turn up the heat, they become more energetic and their orientations mix, putting them in a new state. Likewise, a quantum system has different states depending on its energy levels.

Marcolli and Cornelissen realized that the set of functions that describe these equilibrium states could also be used to characterize their number fields. Such unexpected connections fascinate Marcolli. Who would have thought that some aspect of number theory, so far removed from the real world, would somehow be related to the quantum states of particles?

"This is an example of using ideas and methods from physics to answer a question that is purely mathematical," she says. "When you formulate the question, there's no physics in it. But the way that you think about it, and how you prove it, uses a lot of physics. It's an interaction between mathematics and physics that is less intuitive than the traditional way we use mathematics in physics problems. This is using physics for problems in mathematics."

SOME SPECTRAL ACTION

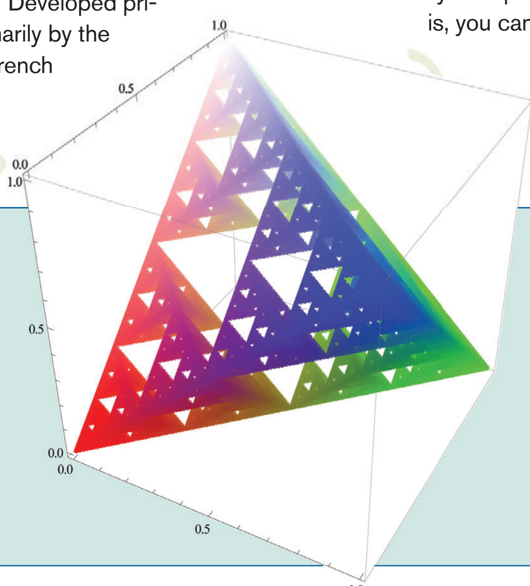
When Marcolli is driving in the more traditional direction on the physics-math highway, using mathematics for problems in physics, she's developing new mathematical models for gravity and elementary particles, with the goal of introducing new ideas for a possible theory of quantum gravity. This is the type of theory that has been touted as a potential theory of everything, combining gravity with the other fundamental forces of nature. A viable theory of quantum gravity has been elusive because it tries to blend two divergent yet wildly successful theories. At one end, there's quantum mechanics, which deals with subatomic particles and tiny scales. At the other end, there's general relativity, which describes gravity and the nature of the universe at cosmic scales. According to general relativity, space is smooth. But when you zoom in to the minuscule scales of quantum gravity—a hundred billion billion times smaller than an electron—space is intrinsically not smooth, with particles popping in and out of existence in what's been dubbed quantum foam. "These two things seem at odds with each other," says graduate student Kevin Teh, who works with Marcolli on new mathematical models for cosmological theories.

Mathematically, this "nonsmoothness" manifests itself in a property called noncommutativity. The normal, everyday rules of math are commutative—that is, A × B = B × A. But quantum mechanics isn't exactly normal, and because of the Heisenberg uncertainty principle, which says you can't precisely measure a particle's velocity and position at the same time, the simple commutativity rule breaks down. Incorporating these messy, unsmooth spaces—called noncommutative spaces—with the smooth spaces of general relativity, requires different mathematical techniques. Perhaps not surprisingly, one such approach is called noncommutative geometry.

"The thing is, you can't visualize noncommutative spaces—you fundamentally cannot."

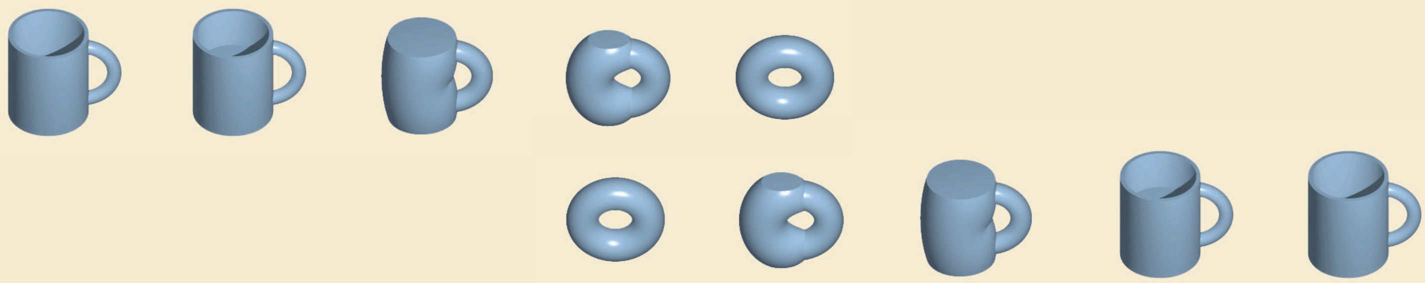
Developed primarily by the French mathematician Alain Connes in the 1980s, noncommutative geometry tries to make sense of these weird spaces, which, in the broader mathematical context, are abstract spaces—not necessarily the space of the universe. "The thing is, you can't visualize noncommutative spaces—you fundamentally cannot," Teh says. But we can come close—the jagged, rough edges of the Mandelbrot set that's popular in calendars and posters are also examples of noncommutative spaces. Or, in a metaphor that gives a better sense of how bizarre these spaces are, Marcolli compares them to the seemingly haphazard arrangement of colors, splotches, and squiggles in a Jackson Pollock painting.

Using noncommutative geometry, Marcolli, Teh, and Elena Pierpaoli at the University of Southern California have



Above: Marcolli and grad student Kevin Teh talk cosmology outside the Red Door Café on campus.

Left: This three-dimensional representation of the solids and voids in a type of fractal called the Sierpinski triangle was calculated by sophomore Christopher Perez, who worked with Marcolli on a Summer Undergraduate Research Fellowships (SURF) project.



recently come up with a model that makes a new prediction about cosmology. This model is derived from something called a spectral action. Generally speaking, an action is a quantity that captures all the relevant physics of a theory into a single, neat mathematical term. The term can then be manipulated mathematically to unzip all the relevant equations.

To visualize a spectral action, Marcolli explains, imagine that the space of the universe is a flat drumhead. Every drum vibrates at certain frequencies depending on its material, size, and shape, giving each instrument a distinct sound. Likewise, the space of the universe has a spectrum of frequencies, and by adding them up in a certain way you generate a spectral action.

Models based on a spectral action are exciting because they're an all-inclusive packaged deal. They give Einstein's equations for general relativity as well as the equations of the Standard Model—the theory of how elementary particles interact—and, as a bonus, equations that describe other observable phenomena. Some versions contain more speculative theories like supersymmetry, which physicists hope to confirm with the Large Hadron Collider (LHC) in Switzerland. The spectral-action model that the researchers came up with provides a mechanism for one of the cornerstones of modern cosmology: inflation, the theory that the newborn universe underwent a rapid expansion in a fraction of a second.

Early theories of inflation have been qualitative, says Teh, describing this cosmic event in broad strokes. But in our era of precision cosmology, in which scientists are making increasingly detailed measurements of the universe, physicists need ever more exact models that make quantitative predictions. As it turns out, the researchers' model predicts that the details of inflation depend on the universe's topology.

While geometry describes an object's specific size and shape, topology studies more general, fundamental features. The classic example involves a doughnut and a coffee mug: because both objects have a hole in the middle—the doughnut hole and the hole that's formed by the mug's handle—both have the same topology. If the doughnut were made out of clay, you could fashion it into a coffee mug while preserving the hole, thereby preserving its topology.

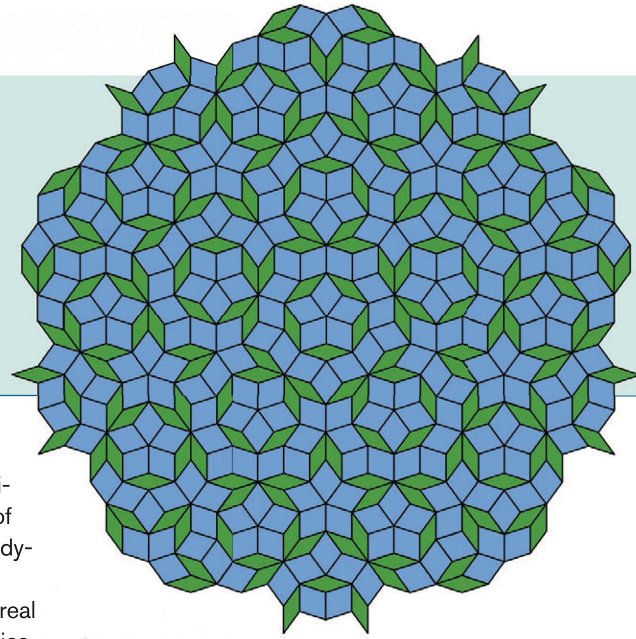
When physicists describe the universe as being flat or nearly flat, they're talking geometry: how space and time are warped according to general relativity. When they talk about whether the cosmos is closed or open, they're referring to its topology. The surface of a sphere, for example, is closed. If an ant walking on a billiard ball takes a straight path, it will eventually retrace its steps. Analogously, in a closed universe, you could shine a flashlight forward and, if you waited long enough, the light would strike you on the back of the head.

"Up to now, we couldn't say much about the topology of the universe," Teh says. But according to the researchers' model, different cosmic topologies would lead to different inflation scenarios, which in turn would leave different signatures on the cosmic microwave background—the pervasive radiation leftover from the Big Bang. This means that, in principle, careful measurements of the cosmic microwave background could reveal what the topology of the universe is like.

Above: Doughnuts and coffee mugs are topologically equivalent because both are pierced by exactly one hole each. In principle, the one could be transformed into the other, which is why one should never delegate a mathematician to make a Starbucks run.

Left: These swirling blots of paint could be seen as an abstract representation of a noncommutative space.

Right: Penrose tilings are also noncommutative spaces. Unlike the tiles on your bathroom floor, which repeat in a simple pattern over and over and over again, a Penrose tiling never completely repeats. You may be able to shift and rotate a Penrose tiling so that a small segment of the pattern overlaps itself, but the rest of it will not.



This is just one in a whole family of models based on the spectral action and noncommutative geometry—and there are many other models that use different mathematical approaches. But it's a good thing that theoretical physicists are rife with ideas, Marcolli notes. With a number of large experiments in cosmology and particle physics now under way—the Planck satellite and the LHC, for example—many of these theories will soon be tested. "It's an ideal time to try new mathematical techniques and see how far you can get by providing testable models," she says.

IT'S COMPLICATED

Noncommutative geometry is an example of pure mathematics having unexpected applications in physics. But whether a new mathematical idea is useful for physics shouldn't be the motivating force for a mathematician. "You cannot plan a priori what will be relevant," Marcolli says. "It's good to develop mathematics because new developments are interesting mathematics. It would be nice if they turn out to be interesting physics as well, but one cannot say whether it will take a few years, a hundred years, or several hundred years for people to discover that some types of mathematics lead to interesting physics." Teh, who considers himself a mathematician, agrees that math should be pursued for the thrill of discovering new ideas. "Mathematicians aren't interested in reworking old ideas," he says. "They're constantly expanding frontiers and finding new things."

On the flip side, Marcolli's work in number theory shows that you never know how physics will repay the favor and give back to math. "Physics is a never-ending source of inspiration for mathematics," Teh says. For example, in the 19th century, Joseph Fourier developed the Fourier series and Fourier transform—basic mathematical tools now used in all branches of physics and engineering—while studying heat flow.

For most of history, there was no real difference between theoretical physics and mathematics, Marcolli says. In the 19th century, as exemplified by the likes of Fourier, mathematicians and physicists were almost indistinguishable. But after the advent of general relativity and quantum mechanics in physics and of similar advances in pure mathematics in the early 20th century, both physics and mathematics became increasingly specialized, and it was then that the relationship status between physics and math became, well, complicated.

Physicists and mathematicians became so caught up in their own subfields that they stopped communicating. Even when they tried, their different languages made it hard for them to understand each other. According to mathematicians, physicists were sloppy, eschewing rigorous proofs for approximations while ignoring the real beauty and truth in pure mathematical ideas. And according to physicists, mathematicians were too enamored with their own thoughts and theorems, which distracted them from the beauty and truth of nature, of what's "real."

But this division was cultural and sociological and had nothing to do with research, Marcolli argues. "It's very artificial in some sense." She points out that even Feynman, for all his teasing of mathematicians, worked with highly sophisticated mathematics. It just became fashionable for both sides to look down on and disassociate from each other, she says. "I think this attitude is very damaging to both physics and math."

Recently, however, she has seen a shift. "The boundary between mathematics and theoretical physics has blurred over the years." Many areas of physics—not just the high-energy physics world of string theory, but also areas such as solid-state physics and quantum information science—have become more theoretical, requiring sophisticated new mathematical tools and forcing both sides to talk again. Marcolli, as someone who's right in the middle, personifies this remerging of the two disciplines. After all, physics and mathematics have the common goal of finding things out, whether it's learning about number fields or cosmic inflation—or just putting out a fire. E&S

Matilde Marcolli received a laurea *in physics from the University of Milan in 1993 and her MS and PhD in mathematics from the University of Chicago in 1994 and 1997, respectively. After a stop at MIT as a C.L.E. Moore Instructor, she received a courtesy appointment at Florida State University, which she still holds today. She was an associate professor at the Max Planck Institute in Bonn, Germany, before becoming a professor of mathematics at Caltech in 2008. Since 2006, she has also held an honorary professorship at Bonn University.*

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