Particles in Motion: The Case of the Loaded Die

by Bradley T. Werner

*"Iacta alea est!"**
— Julius Caesar

PROBLEMS IN THE BASIC SCIENCES can be divided into two categories: those that have been around for a long time and have only been brought to a stage of partial solution, and those that were posed as a result of recent innovations or discoveries. A subset of the former category consists of problems that were investigated in the past but were not solvable by available methods. The development of new tools, particularly modern high-speed computers, has reawakened interest in many such questions. You can discover abandoned problems by venturing to a dark corner of a library and browsing through dusty volumes of a past era, but it often turns out that there is a strong relationship between the original question and current investigations and applications.

A group in the Division of Physics, Mathematics and Astronomy at Caltech (composed of graduate student Gary Gutt, Senior Research Associate Peter Haff, Professor of Physics Tom Tombrello, a passel of undergraduates, and myself — supported by the National Science Foundation) has been studying a problem of this nature: the dynamical behavior of granular materials. A granular material consists of a number of extended objects that interact through very strong compressional forces, as well as through friction, and that move according to Newton's equations for linear motion and Euler's equations for rotational motion. Examples of dynamic granular systems include rock slides, sand dunes, planetary rings, snow avalanches, icebergs in an ice jam, and dry dogfood. Our aim is to derive the general behavior of these systems starting from the level of the properties of individual grains, in a manner similar to the derivation of the kinetic theory of gases and fluids from a consideration of the properties of the basic molecular constituents. The desert environment is the home of many granular systems, and much of our work depends on observations made, ideas and information gathered, experiments performed, and inspiration acquired on trips to the Mojave Desert.

A fundamental problem in grain dynamics is to describe the interaction of a single particle with a flat plane in the presence of a gravitational field. The feeling is inescapable that

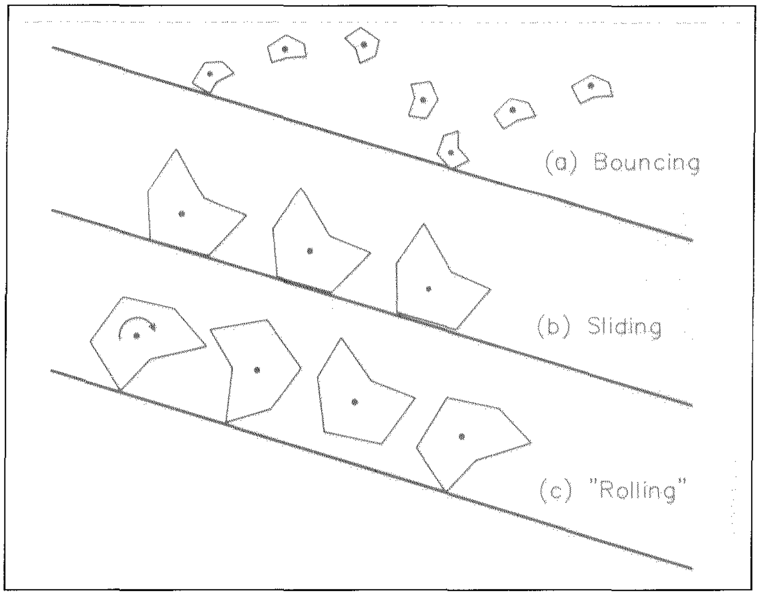
**"The die is cast!"*

if this problem is not amenable to solution, understanding a sand dune, composed of many trillions of grains, is beyond hope. A common reaction to hearing of the single-particle case is that surely the solution may be found in some freshman physics text. In fact, the motion of a single particle on a plane inclined with respect to gravity is far from simple to analyze, particularly if the particle is allowed to have arbitrary shape, and if the collisions between plane and particle include collisional energy loss and friction.

From a simulation study of a two-dimensional particle moving down an inclined plane, I concluded that such particles exhibited three types of motion: sliding, bouncing, and "rolling." As illustrated in the bottom figure at right, an irregularly shaped particle rolls in a rather peculiar manner, with the points farthest out from the center of mass (about which the particle rotates) being the only ones to contact the plane. I found that for sufficiently high friction and energy loss per collision, those particles that went far enough down the plane would eventually move predominantly in this rolling mode, with the bounces perpendicular to the plane being damped out by the energy loss at each collision; particles that engaged in sliding over a significant distance came to rest. Subsequent work by undergraduate Chris Chen on the motion of an ellipse on an inclined plane supports these ideas and highlights the critical importance of the incline angle on the mode of motion. In a practical vein, researchers at the Laboratorio di Fisica Terrestre in Switzerland have used computer simulations to study the motion of rocks that tumble down upon highways in the Alps.

Last year I embarked on a field excursion to investigate the movement of single rocks downslope. In a lonely canyon in the Mojave, just off the Old Spanish Trail, I found a slope covered with rocks ripe for rolling. I scrambled to the top and began sending massive boulders tumbling down the mountainside, with the sharp retorts resulting from rock hitting rock echoing off the canyon walls. Almost every rock that was dislodged either went into the rolling mode or slid rapidly to a stop. When the tumbling rock hit a rock of comparable size or larger, it would be propelled upwards, but the motion perpendicular to the slope would quickly diminish as the rock settled back to rolling. Having exhausted the supply of rocks at the top, I descended the mountain to be pleasantly surprised that the tumbling boulders had implanted "rock-prints" in the sandy wash at the base of the slope, illustrating that these rocks were engaged in rolling.

I was sufficiently inspired by this desert trip to begin the study of single three-dimensional particles. A cube is a simple, yet interesting example. Graduate student Tobi Delbrück suggested that I paint dots on the six faces of the cube and call it a die, and further, that I study the motion of a die that has been altered so that the sides have



Above: The three primary types of motion for a particle moving downslope are: (a) Bouncing — the particle possesses a large amount of energy normal to the plane; this mode of motion usually decays into either sliding or rolling. (b) Sliding — the particle slides without rotating and is often brought to a stop by frictional forces. (c) Rolling — the particle rotates as it moves downslope, with the center of mass staying at roughly a constant height above the plane and only the points farthest from the center of mass contacting the plane.

These "rock prints" were made in a sandy wash in the Mojave Desert by a rock in the rolling mode.

unequal probabilities of landing face-up at the end of a roll: the case of the loaded die.

Being unknowledgable about the world of gambling, I decided to delve into the history of gambling with dice to obtain some background. By far the most authoritative guide is *Scarne on Dice*, written by John Scarne, who saved American soldiers in World War II millions of dollars by teaching them how to detect the tricks of gambling dens. According to Scarne, the ancient civilizations of Egypt, Greece, Rome, and Korea all used dice for gambling. The first written record of modern-appearing dice is found in a 2,000-year-old Sanskrit manuscript. These dice were loaded, suggesting that cheating was present from the start of die history.

Modern dice are cube-shaped, generally three-quarters of an inch on a side, and are made of cellulose. Gambling establishments use "perfect dice," which are square to a tolerance of roughly 1/5,000 of an inch. In contrast, the dice used in my investigation, borrowed from a worn-out Monopoly game, are far from perfect. Yet simply by throwing such a die across one's office floor, it is possible to observe many of the phenomena that make dice interesting. You get to observe a few other things too, including the puzzled stares of passing secretaries and the enthusiastic encouragement of dedicated craps players.

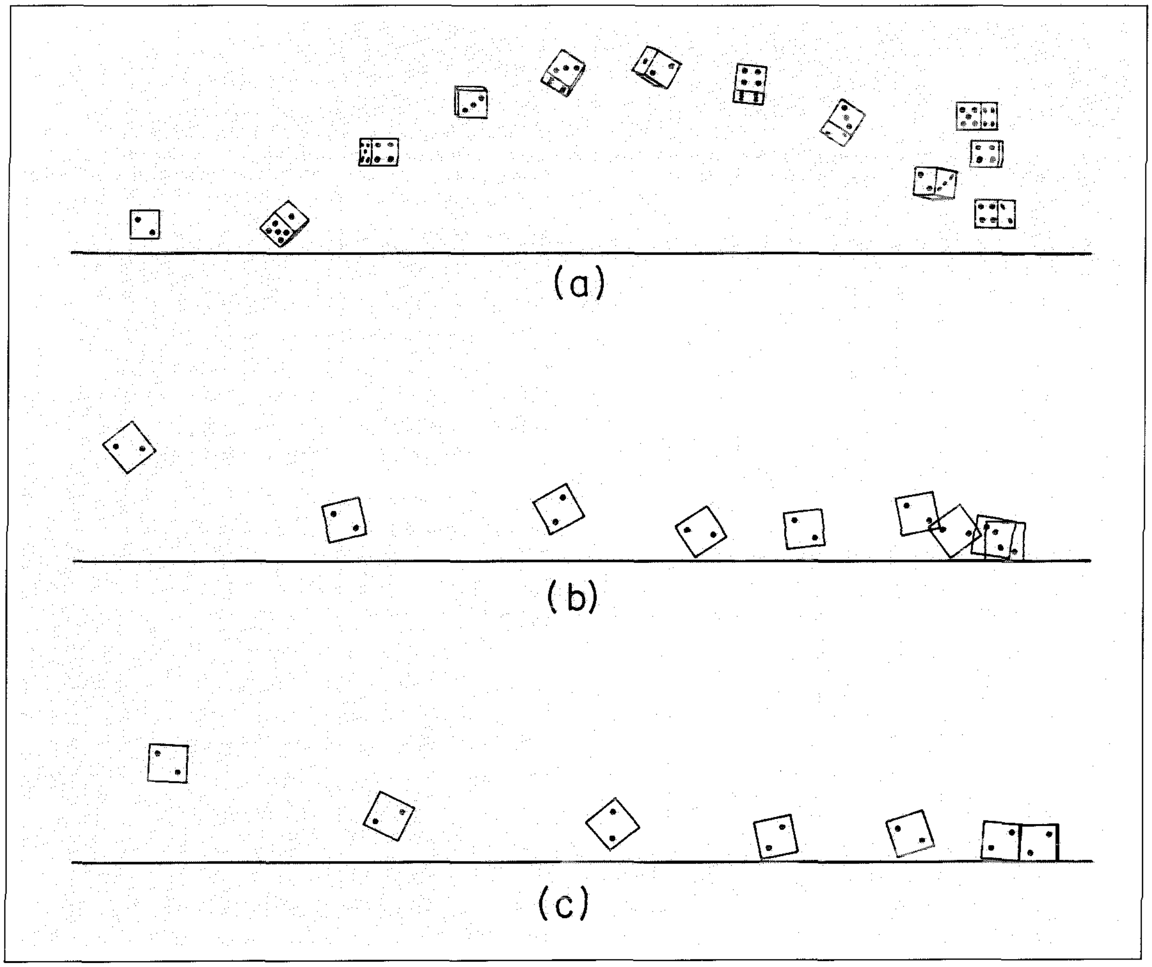
Eventually, when the crowd cleared out, I set about classifying the behavior of the die. It was clear from the start that, at least when interacting with a typical Caltech office floor, the die-floor collisions are rather elastic; that is, only a small fraction of the die's energy is lost in each collision. Related to this is the apparent unpredictability of the value of the die (the number of dots on the top face). When dropped from a distance equal to the die's own height, despite much concentration on keeping the bottom face parallel to the floor, the die will often turn up a different face. This instability makes the value of the thrown die essentially random, and it is no doubt appreciated by casino operators and honest gamblers. Talented cheaters can throw the dice in such a way as to influence the outcome, but such a skill is acquired only through years of practice.

There are a variety of ways to alter a die so that one or more faces have a larger probability of being on top when the die has come to a stop. The most popular method is to change the distribution of mass within the die in order to shift the center of mass away from the center of the cube toward one of the faces, edges, or points. Gambling lore then says that the one, two, or three faces (respectively) closest to the new center of mass will land face down with enhanced probability. With the transparent dice that are currently in common use, the cheating gambler is generally forced to place small weights behind the dots. Casinos detect loaded dice by dropping them carefully in a glass of water; a loaded die will tend to turn with the loaded side down while descending through the fluid, whereas a fair die will fall with little rotation. Scarne states that "Newton doped out the law of gravitation and dice players began to admit that the fall of dice is controlled by gravity rather than by psychic manipulations. . ." My goal is to add some physical detail to Scarne's basically sound explanation for the dynamical behavior of dice.

In my studies of a loaded die in motion, as well as in other granular materials problems, I have found that dynamical simulations are crucial to formulating and verifying a model. Many of the die simulations, including this one, were performed on an Apple MacIntosh; each simulation requires several hours of MacIntosh time. In these simulations, the motion of the die is stepped forward in time according to the laws of Newton and Euler. When one of the eight points on the die begins to penetrate the plane on which it is tumbling, a force is applied, which increases rapidly with the amount of penetration. Frictional forces are also applied at this point. Each collision between the die and the plane results in a loss of a fraction of the die's energy of motion, or kinetic energy.

This simulation picture embodies all of the important characteristics of a real moving die: a nearly rigid particle and energy loss during collisions and friction. I can adjust these characteristics, as well as the mass distribution within the die, at will. In the top figure (a) on the opposite page I show a succession of images of a fair (unloaded) tumbling die.

Using the simulations I have constructed two models for predicting the probability of each side landing face up. These models represent a starting point to which I hope to add refinements as the investigations progress. Both models make the simplification that the die rotates in such a way that only four of the faces can turn up: in other words, it tumbles as if it were a square rather than a cube. Therefore, when the center of mass is located

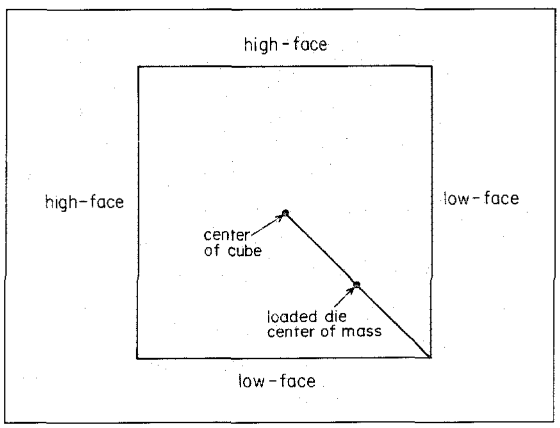


Time-sequenced images of simulated dice moving from left to right: (a) A fair die tumbling on all six sides. (b) A loaded die constrained to tumble on just four sides. (c) A fair die constrained to tumble on just four sides.

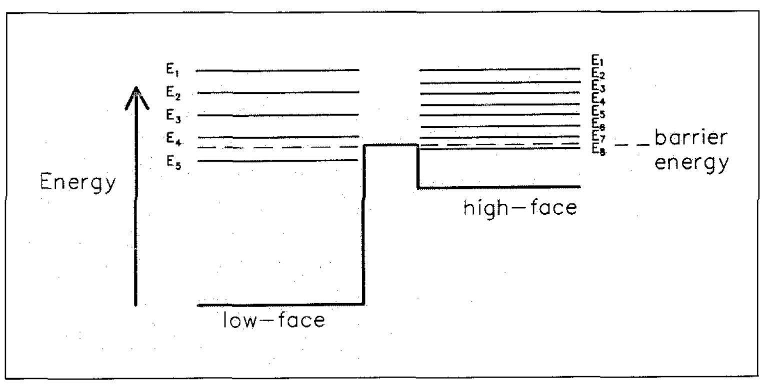
along the diagonal from the center of the cube to an edge (the face diagonal), there are two types of faces: those adjacent to the center of mass, termed "low" faces, and those farthest from the center of mass, or "high" faces. I picture the tumbling die as "progressing" from face to face and continually losing energy until it is finally "captured," with one of the faces resting on the plane.

The first model ignores the order in which faces come up. The idea is schematically illustrated in the bottom figure at right by plotting the energy barrier that must be overcome for a die to transit from a state where a low face is down to a state where a high face is down and vice versa. As the die tumbles around on the plane losing energy and approaching the barrier energy, if it is in a low-face-down state, the behavior of the die is more sensitive to the presence of the barrier, which acts like a brick wall for the die to hurdle over, than if it is in the high-face-down state, where the barrier appears like a mere crack in the sidewalk.

The second model introduces a correlation in the order of the faces hitting the plane; the

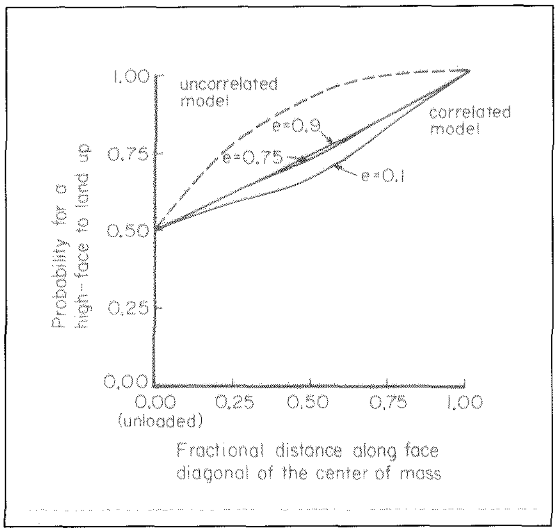


Left: The die in this investigation was loaded in such a way that the center of mass was shifted halfway along the diagonal connecting the center of the cube and the middle of one of the edges (the face diagonal). This creates two types of faces: two low faces, close to the center of mass, and two high faces, far from the center of mass. The models described here ignore the other two faces of the cube.



Below: For the model that does not consider the order in which the die faces can be turned up, a low face can be captured face down much more easily than a high face can.

The probability for a high face to land up is plotted against the fractional distance along the face diagonal of the center of mass. The results for the model that does not consider the order of impact are plotted as a dashed line and are valid for collisions in which most of the energy is retained by the die. The correlated (rolling) model results are plotted as solid lines. These results depend on e, the fraction of the energy retained at each collision.



die is assumed to roll. In the figure above, the predictions of these two models for the probability of a high-face landing up are plotted as a function of the fractional distance along the face diagonal, through which the center of mass has been moved in the process of loading the die. The second model predicts somewhat less advantage to be gained by loading a die, basically because with the enforced correlation there are fewer opportunities to get "captured" in the desired state (a low-face-down state). It is also of interest that when the collisions with the plane become less elastic, the probability of being captured in the low-face-down state decreases. This happens because the system doesn't have the chance to explore all of the available states, in a manner similar to the way in which rapidly quenching a metal from the melt produces an amorphous or unstructured material, whereas slow cooling results in a crystal.

The center of mass of the die was moved to halfway along the diagonal connecting the center of the cube with one of the edges, as shown in the middle figure on page 23. The motions of a die so loaded and of an unloaded die are compared in (b) and (c) respectively at the top of page 23. Note that the motion of the loaded die is slightly more erratic than that of the unloaded die, which is the trend seen in the simulations. Sixteen simulations of this tumbling loaded die started with random initial orientations (but with the constraint that the cube roll as a square) were performed, with the result that for 75 percent of them the die stopped with the high face up. This compares with a prediction of 93 percent for the first model and 72 percent for the second model. The agreement between the simulations and the second model is encouraging.

For the purpose of accurately verifying and refining the second model, it will be necessary to provide better statistical definition of the simulation results, and therefore I expect the MacIntosh to be chugging away at odd hours in the coming months. I hope to be able to consider several additional aspects of a tumbling loaded die in the models, including the effect of the free rotation of a cube (rather than a square). In addition, it would be of interest to investigate other methods of altering dice, including shortening some of the faces and altering the collision characteristics of one or more of the faces. I also plan experiments with loaded dice. Perhaps a few trips to Las Vegas will be necessary to investigate fully all aspects of this complex problem.

But, unlike the group of physics students from UC Santa Cruz who, while betting at a casino, covertly used a computerized shoe to predict where a roulette ball would land, I have no plans to use my knowledge of loaded dice to enhance my meager earnings as a graduate student. The roll of a die, the power of avalanches and rock slides, and the sand-grain ripples on a sand dune have intrigued man for ages. It is wonderful to be alive at a time when a detailed physical understanding of such phenomena appears to be attainable. □