Like Chocolate for String Theory

by Douglas L. Smith

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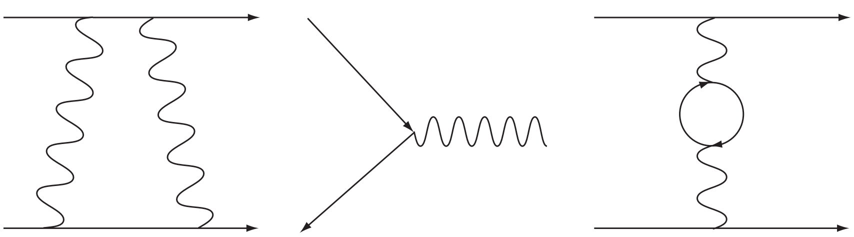


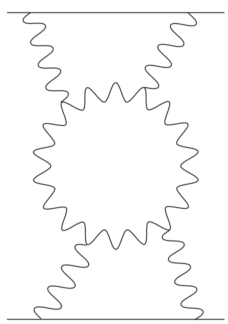
In Feynman diagrams, time moves from left to right. Every line is a particle, and every junction is an interaction. Straight lines are protons, electrons, and the like; wavy lines are force-carriers like photons and gluons. The diagram shows the manner in which a set of particles interact, not their actual directions or speeds. So in the diagram above left, two electrons (arrows) exchange a photon, and then a while later exchange another one. In the center diagram, an electron and a positron (denoted by a backwards arrow, since it is the electron's antiparticle) annihilate one another, producing a photon. And in the right-hand diagram, a photon emitted by an electron produces an electron-positron pair that recombines into a photon before being absorbed by the other electron. As the processes get more and more complex, a picture can truly be worth a thousand words.

Gravity glues galaxies together, while deep within the atom other forces reign supreme. Do galaxies and protons play by the same rules? Professor of Theoretical Physics Hirosi Ooguri and Harvard's Cumrun Vafa, fresh off a six-month visit as a Moore Distinguished Scholar, are trying to find the common ground between the two realms. On the atomic scale, the so-called Standard Model explains three of the universe's four basic forces—electromagnetism, and the strong and weak nuclear forces—in terms of quantum mechanics. And string theory is hot with folks trying to come up with a quantum treatment of gravity and enfold the Standard Model into a "Grand Unified Theory of Everything." The two theories just don't mesh, but Ooguri and Vafa have managed to nudge them into a closer alignment. In the process, they've cleared a mathematical minefield in the Standard Model using techniques they'd developed for working with strings.

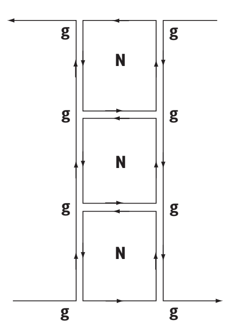
According to the Standard Model, protons and neutrons contain three quarks each. So you'd think that if you hit a proton hard enough you ought to be able to knock one loose, but try as we might, we've never seen a free quark. That's because quarks are held together by the "strong interaction," which increases with distance, so a proton is essentially wrapped in rubber bands. The more you stretch them, the harder they snap back. This strong nuclear force is carried by particles called gluons, the swapping of which makes quarks clingy.

Physicists normally work with such exchanges by drawing little cartoons called Feynman diagrams, showing all the possible things the particles could do. Say you have two electrons. Every now and then, one of them might emit a photon that gets absorbed by the other. In very rare cases, the photon could split in midflight, turning into an electron and a positron, which then recombine to turn back into a photon.

> The harder you pull quarks apart, the more gluons they will exchange as they try to keep their grip. The reason more gluons get exchanged is because the coupling constant grows, and the coupling constant grows because the gluons interact. It's a chicken-and-egg problem.



In the Feynman diagram above, two quarks emit a pair of gluons that then exchange another pair of gluons among themselves. The diagram has eight vertices, labeled g, and three complete loops, labeled N, as shown below. Its contribution to the overall process is proportional to g^8N^3.



And in extremely rare cases . . . you get the idea. You can calculate each diagram's individual effect, add them all up, and eventually derive an overall description of the particles' behavior. In general, the more complicated the diagram, the less likely the process depicted in that diagram is to happen, so you can cut off the calculation at any level of complexity and get a corresponding level of accuracy. "That's how things work when we apply the Standard Model to high-energy collisions, as shown by Professor of Theoretical Physics David Politzer and others, or to the various precise computations in quantum electrodynamics that Feynman studied so successfully," says Ooguri.

Each diagram is represented by a single term in the expansion, or overall calculation, and every term contains two key parameters. The first, called *g*, is the coupling constant, which is a measure of the strength of the particles' interaction. It's raised to the power of the number of vertices, or places where lines meet, in the diagram. The second, called *N*, is raised to the power of the number of closed loops in the diagram. So, for example, the odds of the Feynman diagram at upper left happening are governed by g^8N^3.

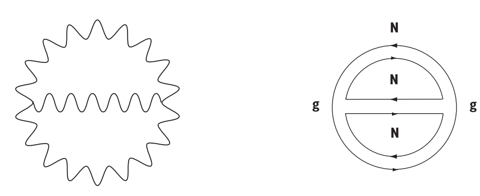
N is always a positive integer, and in the Standard Model, *N* equals three because quarks come in three "colors." More generally, *N* is the rank of the matrix in the SU(*N*) gauge-symmetry group—don't ask: all you need to know is that the Standard Model is a gauge theory. In gauge theories, forces are carried by particles, such as gluons and photons; the elusive quantum-gravity particle is called the graviton.

If you're dealing with electricity, magnetism, or the weak nuclear force, the coupling, *g*, is very small—for electromagnetism at atomic distances, it's about 0.1—and the high-power terms fade rapidly into oblivion. "If each vertex costs you *g*, then the more complicated the diagram becomes, the higher the power of *g* you get, and that suppresses the diagram," Ooguri explains. "So if *g* is small, then you need only worry about the relatively simple Feynman diagrams."

Unfortunately, the harder you pull quarks apart, the more gluons they will exchange as they try to keep their grip. The reason more gluons get exchanged is because the coupling constant grows, and the coupling constant grows because the gluons interact. It's a chicken-and-egg problem. The method gets stood on its head—the more complex the Feynman diagram, the more likely it is to occur. You get stuff that looks like fine French lace, and the calculation spins wildly out of control. So successive terms get bigger and the calculation never settles down on an answer.

But Gerardus 't Hooft, who shared the 1999 Nobel Prize in physics with Martinus Veltman "for elucidating the quantum structure of electroweak interactions," saw a way out. Since the calculation depends on *N* as well as *g*, and *N* is always greater than one, he figured out a way to expand the equations in terms of 1/*N*. You still have to consider all the Feynman diagrams, but now the more complicated the diagram, the better—as you divide by higher and higher powers of *N*, the terms get smaller and smaller.

't Hooft's approach allows you to add up infinitely many Feynman diagrams by classifying them by their topologies rather than their number of vertices. To see what this means, consider the case of three lines meeting at two vertices, like the international "do not" symbol. This diagram



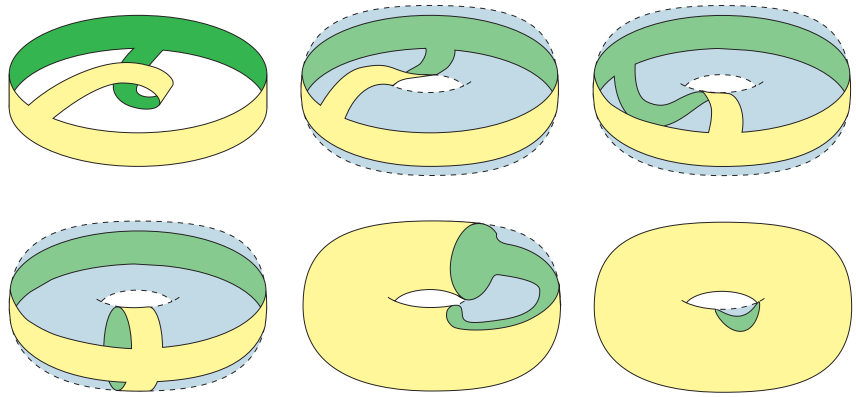
This three-gluon exchange (left) has two vertices and three complete loops (right).

You can make your own one-loop, two-vertex unflat surface. Cut out the three strips at right. Lay them out in a T, green side up, and staple them together. Staple the arms' free ends together, forming a ring that's yellow outside and green inside. Insert the T's leg through the ring from below, bring it out the top, give it a half-twist, and staple it to the front of the ring. If you've done this correctly, you'll have one all-yellow surface and one all-green surface. To show that there's only one edge, run a marking pen along it—you can color all the edges and return to the starting point without lifting the pen.

represents a "vacuum" exchange of three gluons—in other words, a triple-gluon swap between two particles that aren't there; in quantum mechanics, empty space is filled with "virtual" particles that pop into being from nothingness and promptly disappear again. The diagram's two vertices give you g^2, and there are three closed loops for N^3. And if you think of the diagram as being made of flat strips, so that each loop is an edge, you get a disk with two holes in it. So far, so good—but now if you take the central strip, give it a half-twist and connect it to the outer edge of the circle instead of the inner one, the new disk will have a single, continuous edge. (Without going into details, the half-twist can happen because *N* is related to the colors of the quarks.) The two vertices remain, but now there's only one loop, for g^2N, as you can prove to yourself by using the strips of paper at right. You can't draw this up-and-over diagram on a sheet of paper, but you can on the surface of a donut, as we will discover. Mathematicians would say that the two disks have different topologies.

A donut is topologically equivalent to a coffee mug because each has one loop. If you stood the donut on edge and very carefully dimpled it with your thumbs, you'd create a depression that could hold coffee, albeit briefly.

Topology, or rubber-sheet geometry, deals with the invariant properties of objects—things that don't change when the object itself is stretched, bent, or otherwise distorted; poking holes or tearing off pieces is not allowed. Thus a donut is topologically equivalent to a coffee mug because each has one loop. If you stood the donut on edge and very carefully dimpled it with your thumbs, you'd create a depression that could hold coffee, albeit briefly. Our twisted "do-not" symbol is equivalent to a somewhat different mug—one



Above: You can transform the twisted "do-not" symbol into a bitten-out donut shell by gently stretching and deforming it. You start by bringing the far end of the center strip around to its near end, forming a loop that encircles the donut like a cigar band. Then stretch the horizontal and vertical loops until they cover most of the surface, leaving one small hole.

with a hollow handle that's open to the mug's interior. In other words, if you filled this cup with piping-hot coffee, it would go up inside the handle as well. This could be a popular design in Alaska, but there's a large finger-burning, lap-scalding lawsuit potential in the Lower 48. And the twisted "do-not" donut is equally unsatisfactory—imagine a chocolate-shelled donut from which a bite has been taken and the donut itself scraped out, so that only the chocolate remains. Homer Simpson would not be happy.

His daughter Lisa would be ecstatic, however, because that's how you draw a twisted disk on a donut. The intact chocolate shell is the donut's surface, and the bitten-into shell is the drawing on that surface of the twisted "do not" symbol. In fact, any Feynman diagram can be drawn on a shell made from the right number of donuts. Picture a whole bunch of them, some perhaps standing on edge, possibly in a big, jumbled pile, all touching one other and completely drenched in quick-hardening chocolate. After the scraping-out, you'd get a hollow shell that looks like one of Henry Moore's sculptures. (Particles that enter or leave the diagram are represented by open-ended tubes—half-eaten donuts—sticking out from the shell.) In 't Hooft's formulation, if you start with n donuts, any diagram drawn on—or bitten out of—that shell comes with a factor of $1/N^{-2n}$. "The number of donuts is a topological invariant," says Ooguri, "and the power of N keeps track of it."

Well, then, why not forget about Feynman diagrams altogether and recast the Standard Model as a theory of chocolate shells? Ooguri and Vafa have shown this is indeed possible—not for the Standard Model itself, not yet—but for a large class of supersymmetric gauge theories in four dimensions. (Remember, gauge theories describe forces in terms of particles; supersymmetry is something required to explain why most particles have mass.) Ooguri and Vafa adapted the language of string theory to describe the donut shells, and it works very well.

String theory had been rescued from obscurity in 1974, when John Schwarz, now the Brown Professor of Theoretical Physics, and his collaborator, the late Joel Scherk of the Ecole Normale Supérieure in Paris, realized that it could be a candidate for the long-sought Theory of Everything. (It had originally been invented for an altogether different purpose that didn't work out, but that's another story.) But while it handled quantum gravity quite nicely, it predicted a universe that didn't match ours in one important respect. Explains Ooguri, "Nature is not symmetric under the exchange of left and right. The world in the mirror is not the same as our world." This effect, known as parity violation, could not be reproduced—the string-theory universe remained stubbornly ambidextrous. Undaunted, Schwarz kept plugging away almost single-handedly until 1984, when he and Michael Green (then at the University of London, now at the University of Cambridge) found the fix that kept the theory internally consistent while allowing parity to be violated. The field took off, and nowadays you can't pick up a popular-science magazine without reading about superstrings, 10- or 11-dimensional universes, M theory, branes, and the like.

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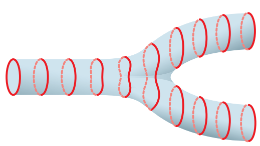
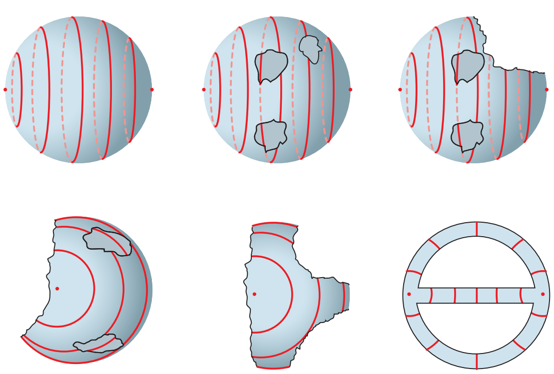
Strings can be thought of as flexible Os. As time passes, a string sweeps out a "world sheet," as shown at left. If the string is moving, the sheet—a cylinder, really—leans in the direction of motion. If the string emits another string, the cylinder forks. As more strings interact, their collective world sheet becomes a network of fused donut shells.

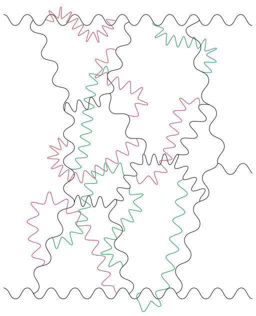


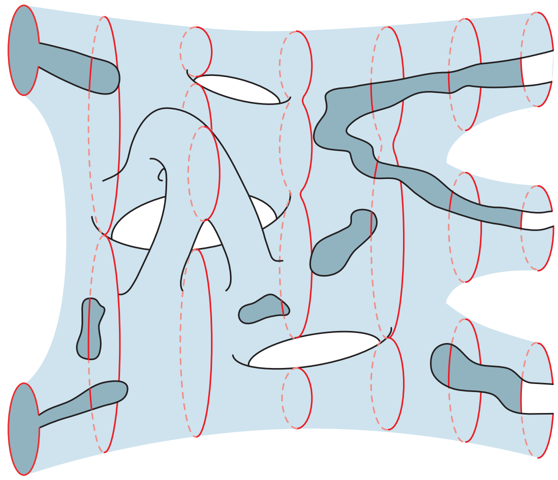
Top left: As in a Feynman diagram, time moves from left to right. Here a string (red) emits another string, causing the world sheet (gray) to fork.



Left: If a string comes into existence briefly and then vanishes, its world sheet is a sphere. Ooguri's and Vafa's exotic domains tear the sphere's surface open, and by stretching three openings in just the right way, you can get the flat disk with two holes. The bottom two spheres have been rotated to show how one hole engulfs nearly an entire hemisphere before the flattening.

But it's not enough for the world sheet to look like a shell. It has to taste like chocolate, or in this case it has to reproduce the adding-up of the Feynman diagrams. Ooguri and Vafa have shown that one particular variant of string theory does just that. Says Ooguri, "When we did the computations, the world sheet started generating some exotic domains because of its internal dynamics. It tore open here and there to create a new phase in which space-time decayed into nothing." Such behavior tends to be the death of theories, as the math generally breaks down, but Ooguri and Vafa were thunderstruck to discover that the strings stayed in the sheet's normal regions, flowing around the exotic domains like water around rocks in midstream. That is, the strings developed gaps as needed to avoid entering these uncharted zones, and then magically closed up again when the danger had passed. "It turns out that this corresponds exactly to a Feynman-diagram computation. The exotic domains create holes in the world sheet, and if you throw them out, you recover the computation from gauge theory. This provides a way to generate open strings out of closed strings, and once you have open strings, you almost have a gauge theory."





Above: If you make a world sheet from enough donuts, you can reproduce any Feynman diagram, no matter how complicated. The Feynman diagram at left has 30 vertices and a tangle of gluons. The red lines are half-twisted paths that rise up out of the page, while the green lines are half-twisted ones that hang under the page. This diagram can be drawn on the five-donut surface at right. (The forked bridge, when squashed flat, becomes three donuts.)

Ooguri and Vafa were working in four dimensions; the current universe-explaining superstring theory operates in 10. (The other six are curled up on themselves, so we don't experience them.) Ahead lies the job of twiddling with those other six dimensions until the Standard Model comes tumbling out. The clincher will come when the calculations predict the masses of the proton, neutron, and so on that are actually observed, and to the same level of precision.

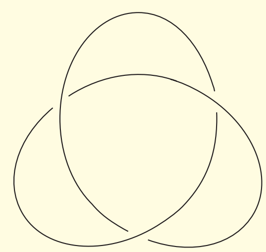
"There's already a string theory that approximates the strong interaction pretty well, but it's not exact," says Ooguri. In this regard, the string theorists are in the same boat as everybody else. Because the Feynman diagrams are so intractable, the other folks have resorted to something called lattice gauge theory, in which space-time is divided into a finite set of points, called a lattice. Then a computer calculates all the fields at each lattice point. Says Ooguri, "The technique has gotten to the point where we can compute particle masses fairly well. But it is not very illuminating.

"We want to do much better. By transforming the calculations into string-theory problems, the techniques Vafa and I, and other collaborators, have worked out over the last 10 years give us a way to compute various quantities *exactly* for a large class of gauge theories. These are calculations we couldn't even approach before, and that's very exciting."

To date, nobody has found a general analytical method capable of handling the strong interaction. In fact, it's such a tough nut to crack that the Clay Mathematics Institute has named it one of seven "Millennium Problems," and has offered a million bucks to the person or persons who succeed. And while the money would be nice, "if we get a handle on this," Ooguri says, "we'll surely learn tons of new things about gauge theory. That's our aim." □

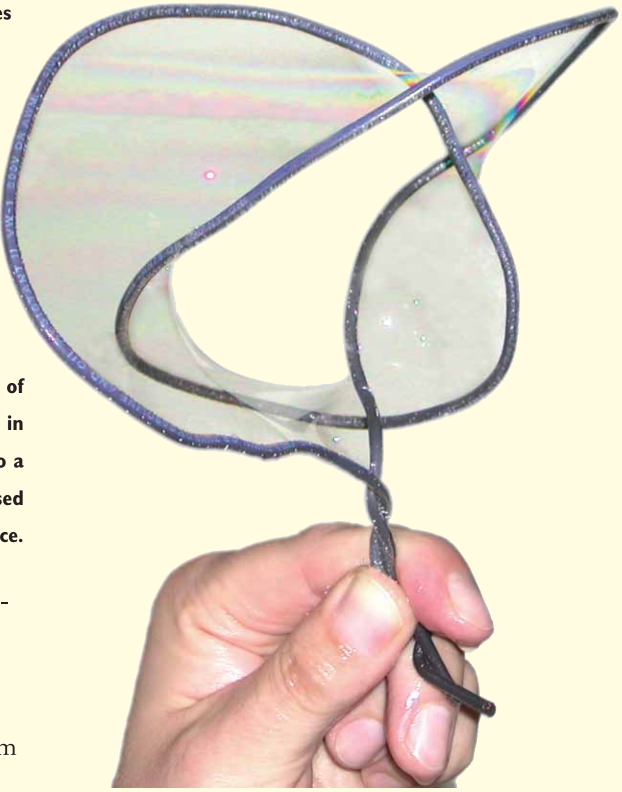
The Number of the Knot

The work also has mathematical applications, particularly in three-dimensional knot theory. A knot can be thought of as a length of rope with its two ends attached to each other. The simplest knot is a circle or ellipse, the so-called unknot; in nontrivial knots the line is wrapped around itself. So in the next-simplest knot, you cross the line over and under itself once, as if you were preparing to tie your shoelaces, before you join the ends. This is called a trefoil knot. And truly complicated knots have loops stuffed through other loops and lines twisted around themselves like the Gawd-awful tangle that that 150-foot, bright-orange outdoor extension cord in your garage is in.



One of knot theory's fundamental problems is to determine whether one knot is equivalent to another—whether the one can be transformed into the other without forcing the line to pass through itself like a magician's linking rings. Mathematicians eventually hope to be able to classify all knots in this manner. A related question is that of deciding whether a given knot is trivial, that is, if it can be disentangled into a circle. Say you have a flat loop of rope—a very long, thin oval. If you treat the tips of the oval like the ends of an ordinary piece of cordage, you can tie the doubled-up rope into additional knots. The result sure doesn't look trivial, but it is—you can get back to the original oval without cutting and splicing anything.

In their quest to classify knots, mathematicians have come up with several invariants, or mathematical expressions that remain unchanged as you pry the knot's loops apart. If the invariants for two knots are different, then, clearly, so are the knots. But nobody has yet come up with a



Surface tension drives soap films to span the minimum possible area, so here's the minimum surface of a trefoil knot, wire-loop and soap-film style. The saddle-shaped surface curves gracefully to connect adjoining turns of the wire, leaving a void in the middle analogous to a donut hole on a closed surface.

formulation for a "complete" invariant—a formulation that says that two knots must be the same if their invariants are the same.

One nearly complete class of knot invariants is called the Jones polynomials, discovered by UC Berkeley's Vaughn Jones. This work won him the Fields Medal, often called the Nobel Prize of mathematics, in 1990. Says Ooguri, "Jones's work initiated a proliferation of knot invariants in the 1980s. Unfortunately, these invariants have not provided much insight into knot theory itself. In particular, the relationships between these invariants and the intrinsic geometric properties of the knots remain obscure."

But, he adds, "while we were trying to figure out the equivalence between gauge theory and string theory, and the physical consequences of that equivalence, we came up with a surprising prediction: for every knot, you can extract an infinite set of integers from the Jones invariant and its generalizations, and these integers have clear geometric meaning. Mathematicians like integers. They think integers are more noble than real numbers. So when we found integers in an unexpected place, it got their attention." In fact, some aspects of Ooguri and Vafa's conjecture have already been proven mathematically.

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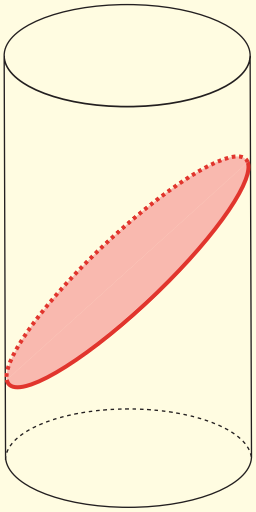
The conjecture arose by analogy to 't Hooft's method for adding up Feynman diagrams drawn on chocolate surfaces. A knot is a one-dimensional object, but it's embedded in three-dimensional space. So Ooguri and Vafa added two dimensions to the knot to make it 3-D, and then placed this 3-D knot in six-dimensional space—six-dimensional because they were trying to work out what happens in those six extra dimensions that come along with string theory. Says Ooguri, "We then looked for minimum surfaces—surfaces of minimum area, like a soap film on a wire loop—that are bounded by the 3-D knot." That's pretty mind-bending, but it's easier to follow in fewer dimensions. For example, take a cylinder and slice it on the bias to make an oval. This oval is an unknot. If you extend the unknot into two dimensions, you get the cylindrical surface. And the unknot's minimum surface in three-dimensional space is the diagonal disk that lies within the cylinder and whose edge is the oval. Moving up the food chain, a trefoil knot can have a fluted minimum surface with a donutlike hole in the center.



An unknot (red) extended into two dimensions creates a cylindrical surface. The unknot's minimum, or soap-bubble, surface in the three-dimensional space containing the cylinder is the slanted ellipse shown in pink.

"The surfaces come with various topologies," says Ooguri, "so we count up the number of surfaces in each topological class. There are infinitely many topological classes—basically the number of donuts again—so we have infinitely many integers." (Of course, a lot of those integers can be zero.) "And the way you count them has close ties to other branches of mathematics, so I hope that insights from those branches will give fresh perspectives to problems in three-dimensional topology." —*DS*